Ordinary Differential Equations

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Syllabus

The syllabus contains the following articles:

- First Order Differential Equation
  - Leibnitz linear equation
  - Bernoulli's equation
  - Exact differential equation
  - Equations not of first degree
    - Equation solvable for p
    - Equation solvable for x
    - Equation solvable for y
  - Clairaut's equation
- Higher Order Differential Equation
  - Second order linear differential equations with variable coefficients
  - Method of variation of parameters
  - Power series solutions

Leibnitz linear equation

Definition

An equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are either constants or functions of x only is called Leibnitz linear equation.
Alternately, the equation may be of the form $\frac{dx}{dy} + Px = Q$, where P and Q are either constants or functions of y only.

Solution

This equation is solved by evaluating the Integration Factor that is given by $IF = e^{\int Pdx}$ and the solution is obtained by $y(IF) = \int Q(IF)dx + c$ for the former case and for the latter x is replaced by y in the IF and the solution.

Questions

- $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$
- $x \log x \dfrac{dy}{dx} + y = 2 \log x$
- $\dfrac{dy}{dx} + y \cot x = 5e^{\cos x}$
- $\dfrac{dy}{dx} = \dfrac{y}{2y \log y + y - x}$
- $\sqrt{1 - y^2} dx = (\sin^{-1} y - x) dy$

Bernoulli's Equation

Definition

An equation of the form $\dfrac{dy}{dx} + Py = Qy^n$, where P and Q are either constants or functions of x only is called Bernoulli's equation.
Alternately, the equation may also be written as $\dfrac{dx}{dy} + Px = Qx^n$, where P and Q are either constants or functions of y only.

Solution

This equation is reduced to Leibnitz linear equation by substituting $y^{1-n} = z$ and differentiating. This generates the Leibnitz equation in z and x that is solved as explained earlier and then z is resubstituted in terms of y. The corresponding changes are made in the latter case of definition.

Questions

- $x\dfrac{dy}{dx} + y = x^3y^6$
- $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$
- $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$
- $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$
- $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

Exact Differential Equation

Definition

An equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be an Exact differntial equation if it can be obtained directly by differentiating the equation $u(x,y) = c$, which is its primitive.
i.e. if

$$du = Mdx + Ndy$$

Necessary and Sufficient Condition

The necessary and sufficient condition for the equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

The solution of $Mdx + Ndy = 0$ is given by

$$\int_{y \text{ constant}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$$

QUESTIONS

- $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$
- $(1 + e^{x/y})dx + \left(1 - \frac{x}{y}\right) e^{x/y} dy = 0$
- $(2xy \cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$
- $xdy + ydx + \frac{xdy - ydx}{x^2 + y^2} = 0$
- $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

Equations reducible to exact equations

Reducible to exact equations

Equations which are not exact can sometimes be made exact after multiplying by a suitable factor (function of x and/or y) called the Integration Factor (IF).

IF by Inspection

- $ydx + xdy = d(xy)$
- $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$
- $\dfrac{xdy - ydx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
- $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right]$
- $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
- $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$
- $\dfrac{ydx + xdy}{xy} = d[\log(xy)]$
- $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

Equations reducible to exact equations

IF for Homoegeneous Equation

If $Mdx + Ndy = 0$ is a Homoegeneous equation in x and y, then $\dfrac{1}{Mx + Ny}$ is an IF provided $Mx + Ny \neq 0$.

IF for $f_1(xy)ydx + f_2(xy)xdy = 0$

For equation of this type, IF is given by $\dfrac{1}{Mx - Ny}$.

Equations reducible to exact equations

IF for $Mdx + Ndy = 0$

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $IF = e^{\int f(x)dx}$.
- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say $g(y)$, then $IF = e^{\int g(y)dy}$.

IF for $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$

In this equation, a, b, c, d, m, n, p, q are all constants and IF is given by $x^h y^k$, where h and k are so chosen that the equation becomes exact after multiplication with IF.

Questions

- $(1+xy)ydx+(1-xy)xdy=0$
- $xdy-ydx=xy^2dx$
- $(xye^{x/y}+y^2)dx-x^2e^{x/y}dy=0$
- $(x^2y^2+xy+1)ydx+(x^2y^2-xy+1)xdy=0$
- $\left(y+\dfrac{y^3}{3}+\dfrac{x^2}{2}\right)dx+\dfrac{1}{4}(x+xy^2)dy=0$
- $(2x^2y-3y^4)dx+(3x^3+2xy^3)dy=0$
- $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$

Equations of first order and higher degree

Definition

A differential equation of the first order and n^{th} degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0, \text{ where } p = \frac{dy}{dx} \tag{1}$$

Equations solvable for p

Resolve equation (1) into n linear factors and solve each of the factors to obtain solution of the given equation.

Questions

- $p^2 - 7p + 12 = 0$
- $xyp^2 - (x^2 + y^2)p + xy = 0$
- $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$
- $p^2 - 2p \sinh x - 1 = 0$
- $4y^2p^2 + 2pxy(3x + 1)3x^3 = 0$

Equations solvable for y

Differentiate equation (1), wrt x, to obtain a differential equation of first order in p and x that has solution of the form $\phi(x, p, c) = 0$. The elimination p from this solution and equation (1) gives the desired solution.

Questions

- $xp^2 - 2yp + ax = 0$
- $y - 2px = \tan^{-1}(xp^2)$
- $x^2 \left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$
- $x - yp = ap^2$

EQUATIONS SOLVABLE FOR x

Differentiate equation (1), wrt y, to obtain a differential equation of first order in p and y that has solution of the form $\phi(y, p, c) = 0$. The elimination p from this solution and equation (1) gives the desired solution.

QUESTIONS

- $y = 3px + 6p^2y^2$
- $p^3 - 4xyp + 8y^2 = 0$
- $y = 2px + p^2y$
- $y^2 \log y = xyp + p^2$

Clairaut's equation

Definition

An equation of the form $y = px + f(p)$ is called Clairaut's equation.

Solution

Differente the equation wrt x, and obtain the solution by putting $p = c$ in the given equation.

Questions

- $y = xp + \dfrac{a}{p}$
- $y = px + \sqrt{a^2p^2 + b^2}$
- $p = \sin(y - px)$
- $p = \log(px - y)$

Linear Differential Equations

Definition

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \tag{2}$$

Linear Differential Equations

Complementary Function (CF)

- If all the roots of equation (2) are real and distint, CF is given by $y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$
- If two roots are equal, say $m_1 = m_2$, then CF is given by $y = (c_1x + c_2)e^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$
- If two roots are imaginary, say $m_1 = \alpha + \iota\beta,\ m_2 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$
- It two pairs of imaginary roots are equal, say $m_1 = m_2 = \alpha + \iota\beta,\ m_3 = m_4 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}[(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x] + c_5e^{m_5x} + \cdots + c_ne^{m_nx}$

Linear Differential Equations

Particular Integral (PI)

- If $X = e^{ax}$, then PI is given by $y = \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$, provided $f(a) \neq 0$.
- If $X = \sin(ax+b)$ or $\cos(ax+b)$, then PI is given by
  $y = \dfrac{1}{f(D^2)} \sin(ax+b) = \dfrac{1}{f(-a^2)} \sin(ax+b)$. Likewise for $\cos(ax+b)$.
- If $X = x^m$, where m is a positive integer, then PI is given by $y = \dfrac{1}{(D)} x^m$.

  Take out the lowest degree term from $f(D)$ to make the first term unity and then shift the remaining term to numerator and apply Binomial expansion upto D^m. Operate term by term on x^m.
- If $X = e^{ax}V$, where V is a function of x, then PI is given by
  $y = \dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D+a)} V$.
- If X is any other function of x, then PI is obtained by resolving the $f(D)$ into linear factors and applying $\dfrac{1}{D-a} X = e^{ax} \displaystyle\int e^{-ax} X dx$

Questions

- $(D^2 + 4D + 5)y = -2\cosh x$
- $(D2 - 4D + 3)y = \sin 3x \cos 2x$
- $(D^2 + 4)y = e^x + \sin 2x$
- $(D^2 + D)y = x^2 + 2x + 4$
- $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$
- $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$
- $(D^2 - 1)y = x\sin x + (1 + x^2)e^x$
- $(D - 1)^2(D + 1)^2 y = \sin^2 \frac{x}{2} + e^x + x$

Cauchy's Homogeneous Equation

Definition

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \qquad (3)$$

where a_is are constants and X is a function of x is called Cauchy's Homegeneous Linear Equation.

Solution

The equation is reduced to an LDE with constant coefficients by putting $z = e^x$ thereby generating an LDE in x and z that can be solved as explained earlier and finally the solution of equation (3) is obtained by putting $z = \log x$.

Questions

- $x^2\dfrac{d^2y}{dx^2}+9x\dfrac{dy}{dx}-25y=50$
- $x^4\dfrac{d^3y}{dx^3}+2x^3\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=1$
- $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=\dfrac{12\log x}{x^2}$
- $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+y=\log x\dfrac{\sin(\log x)+1}{x}$

Legendre's Linear Equation

Definition

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X \quad (4)$$

where a_is, a and b are constants and X is a function of x is called Legendre's Linear Equation.

Solution

The equation is reduced to an LDE with constant coefficients by putting $a+bx = e^z$ thereby generating an LDE in x and z that can be solved as explained earlier and finally the solution of equation (4) is obtained by putting $z = \log(a+bx)$.

QUESTIONS

- $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos\log(1+x)$
- $(1+2x)^2\dfrac{d^2y}{dx^2}-6(1+2x)\dfrac{dy}{dx}+16y=8(1+2x)^2$
- $(3+2x)^2\dfrac{d^2y}{dx^2}-2(3+2x)\dfrac{dy}{dx}-12y=6x$

Variation of Parameters

This method is applicable for the second order differential equation of the form $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = X$
Let the CF of this equation be

$$y = c_1y_1 + c_2y_2$$

. Then the PI of this equation is given by

$$y = uy_1 + vy_2$$

where

$$u = -\int \frac{y_2X}{W}dx$$

and

$$v = \int \frac{y_1X}{W}dx$$

where W is the Wronskian of $y_1,\ y_2$.

Questions

- $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$
- $\dfrac{d^2y}{dx^2} + y = cosec\ x$
- $\dfrac{d^2y}{dx^2} + y = x\sin x$
- $y'' - 2y' + 2y = e^x \tan x$

Series Solution

We discuss the method of solving equations of the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{5}$$

where $P_0(x), P_1(x)$ and $P_2(x)$ are polynomials in x, in terms of infinite convergent series.

Solution

Divide equation (5) by $P_0(x)$ to get

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{6}$$

where $p(x) = \dfrac{P_1(x)}{P_0(x)}$ and $q(x) = \dfrac{P_2(x)}{P_0(x)}$

Series Solution

Ordinary Point

$x = 0$ is called an ordinary point of equation (5) if $P_0(0) \neq 0$.
In this casem the solution of equation (5), can be expressed as

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Singular Point

$x = 0$ is called a singular point of equation (5), if $P_0(0) = 0$.
In this case, the solution of equation (5) can be expressed as

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Solution when $x = 0$ is an ordinary point

Solution

Let $y = \sum_{k=0}^{\infty} a_k x^k$ be the solution of equation (5). Then, on differentiating

$$\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

1. Substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (5).
2. Equate to zero the coefficients of various powers of x and find $a_2, a_3, a_4, \ldots$ in terms of a_0 and a_1.
3. Equate to zero the coefficient of x^n. The relation so obtained is called the recurrence relation.
4. Give different values to n in the recurrence relation to determine various a_is in terms of a_0 and a_1.
5. Substitute the values in the above mentioned series to obtain the solution with a_0 and a_1 as arbitrary constants.

QUESTIONS

- $\dfrac{d^2y}{dx^2} + xy = 0$
- $y'' - xy' + x^2y = 0$
- $(2 - x^2)y'' + 2xy' - 2y = 0$

Solution when $x = 0$ is a regular singular point I

Let $y = \sum_{k=0}^{\infty} a_k x^{m+k}$ be the solution of equation (5). Then, on differentiating

$$\frac{dy}{dx} = \sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}.$$

1. Substitute the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in equation (5).
2. Equate to zero the coefficients of lowest powers of x. This gives a quadratic equation in m, which in known as indicial equation.
3. Equate to zero the coefficients of other powers of x to find $a_1, a_2, a_3, a_4, \ldots$ in terms of a_0.
4. Substitute the values of $a_1, a_2, a_3, \ldots$ in above said solutionto get the series solution of (5) having a_0 as the arbitrary constant. Though, it is not the complete solution as the same should have two arbitrary constants.
5. The method of complete solution depends on the nature of roots of the indicial equation.

Solution when $x = 0$ is a regular singular point II

Case I When the roots m_1, m_2 are distinct and not differing by an integer. Then the complete solution is given by

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

Case II When the roots m_1, m_2 are equal. Then the complete solution is given by

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Case III When the roots $m_1 < M_2$ are distinct and differ by an integer. Then th ecomplete solution is given by

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Questions

- $2x^2\dfrac{d^2y}{dx^2} + (2x^2 - x)\dfrac{dy}{dx} + y = 0$
- $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 4)y = 0$
- $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0$

Functions of Complex Variables

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Complex Function

Definition (Complex Function)

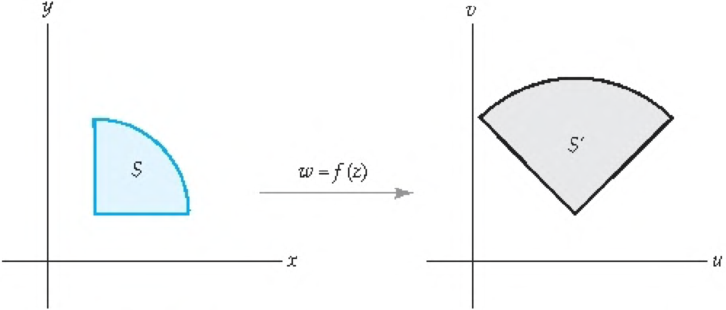
A Complex Function is a function f whose domain and range are subsets of the set $\mathbb{C}$ of complex numbers.

Complex Functions as Mappings

- Graphs are used extensively to investigate the properties of real functions. However, the graph of a complex function lies in four-dimensional space and so we cannot use the graphs to study complex functions.
- So we discuss the concept of complex mapping.
- Every complex function describes a correspondence between points in two copies of the complex plane. Specifically, the point z in the $z-$plane is associated with the unique point $w = f(z)$ in the w-plane.
- The term complex mapping is used in place of complex functions when considering the function as this correspondence between points in the z-plane and the points in the w-plane.
- The geometric representation of a complex function $w = f(z)$ consists of two figures: the first, a subset S of the points in the z-plane and, the second, the set S' of the images of the points in S under $w = f(z)$ in the w-plane.

Complex Functions as Mappings

If $w = f(z)$ is a complex function, then both z and w lie in a complex plane. Moreover, treating the complex numbers as two-tuple points say (x, y) and (u, v) respectively for z and w, then the complex mapping maps the point $z = (x, y)$ of the $x - y$ plane into the point $w = (u, v)$ of the $u - v$ plane.



(a) The set S in the z-plane

(b) The image of S in the w-plane

Example - z^2

Now $w = f(z) = z^2 = (x + \iota y)^2$
Thus,

$$f(x, y) = (x^2 - y^2, 2xy)$$

Thus the function $f(z) = z^2$ is equivalent to the real system of equations given by

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}$$

Complex Limits

- A complex limit is, in essence, the same as a real limit except that it is based on a notion of "close" in the complex plane. Because the distance in the complex plane between two points z_1 and z_2 is given by the modulus of the difference of z_1 and z_2, the precise defintion of a complex limit will involve $z_2 - z_1$.
- The phrase "$f(z)$ can be made arbitrarily close to the complex number L" can be stated precisely as: *for every* $\epsilon > 0$, z *can be so chosen that* $|f(z) - L| < \epsilon$.
- Since the modulus of a complex number is a real number, both ϵ and δ represent small positive real numbers in the following definition of complex limit.

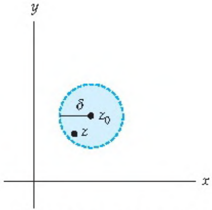
Complex Limits

Limit of a Complex Function

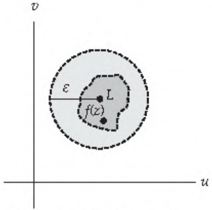
Suppose that a complex function $w = f(z)$ is defined in a deleted neighbourhood of z_0 and suppose that L is a complex number. The limit of f as z tends to z_0 exists and is equal to L, written as $\lim_{z \to z_0} f(z) = L$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

That is, f maps the deleted neighbourhood $0 < |z - z_0| < \delta$ in the z-plane into the neighbourhood $|w - L| < \epsilon$ in the w-plane.

Geometrical Interpretation



(a) Deleted δ-neighborhood of z_0



(b) ε-neighborhood of L

In figure (a), the deleted neighbourhood of z_0 shown in colour is mapped onto the set shown in dark gray in figure (b).

As required by the definition, the image lies within the ϵ-neighbourhood of L shown in light gray in figure (b).

Complex Limits

In particular, if we write

$$\begin{aligned} f(z) &= u(x,y) + \iota v(x,y) \\ L &= L_1 + \iota L_2 \\ z_0 &= x_0 + \iota y_0 \end{aligned}$$

then $\lim\limits_{z\to z_0} f(z) = L \;\Leftrightarrow\; \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = L_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = L_2$

Thus, the limit of a complex function can be viewed as a system of limits of real functions.

Example - limit does not exist

Example Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.
Consider two different paths of letting z approach 0.

Continuity of complex functions

A complex function f is said to be continuous at a point z_0 if the limit of f as z approaches z_0 exists and is same as the value of f at z_0. That is,

Continuity of complex function

A complex function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Analogous to the real functions, if a complex function is continuous, the following three conditions must be met.

- f is defined at z_0
- the limit $\lim_{z \to z_0} f(z)$ exists
- $\lim_{z \to z_0} f(z) = f(z_0)$

Example-continuity

Example Check the continuity of the function $f(z) = z^2 - \iota z + 2$ at the point $z_0 = 1 - \iota$.

Differentiability and Analyticity

Suppose $z = x + \iota y$ and $z_0 = x_0 + \iota y_0$; then the change in z_0 is the difference $\Delta z = z - z_0$ or $\Delta z = (x - x_0) + \iota(y - y_0) = \Delta x + \iota \Delta y$. If a complex function $w = f(z)$ is defined at z and z_0, then the corresponding change in the function is the difference $\Delta w = f(z + z_0) - f(z_0)$. The derivative of the function f is defined in terms of a limit of the difference quotient $\Delta w / \Delta z$ as $\Delta z \to 0$.

Derivative of a complex function

Suppose the complex function f is defined in a neighbourhood of a point z_0. The derivative of f at z_0, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided the limit exists regardless how Δz approaches 0. This implies that in complex analysis, the requirement of differentiability of a function $f(z)$ at a point z_0 is a far greater demand than in real calculus of functions $f(x)$ where we can approach a real number x_0 on the number line from only two directions.

If a complex function is made up by specifying its real and imaginary parts u and v, such as $f(z) = x + 4\iota y$, there is a good chance that it is not differentiable.

Derivative

If $w = f(z) = u(x,y) + \iota v(x,y)$ then

$$\begin{aligned} f'(z_0) &= \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left[\frac{\Delta u + \iota \Delta v}{\Delta x + \iota \Delta y} \right] \end{aligned} \tag{1}$$

- The limit above is independent of the path in which Δz tends to zero.
- This is much more stronger than the concept of directional derivative of the real-valued function of several real variables.
- If the directional derivatives were to exist in every direction, all that is needed is that in each direction the limit exists and that the limit could be different in different directions.
- However, in the definition above, the limit must exist no matter what direction Δz approaches zero and the value of the limit is same.

Derivative of a Complex Function

Now the emphasis is that not only the limit as mentioned in eq. (1) exists but be the same in all the directions and that imposes some strong conditions on Δu and Δv.

Lets talk in particular about two directions.

Case I If $\Delta y \equiv 0$ (y is constant). Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} + \iota \frac{\Delta v}{\Delta x} \right] \\ &= \left[\frac{\partial u}{\partial x} + \iota \frac{\partial v}{\partial x} \right]_{z_0=(x_0,y_0)} \end{aligned}$$

Case II If $\Delta x \equiv 0$ (x is constant). Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \left[\frac{\Delta u}{\iota \Delta y} + \iota \frac{\Delta v}{\iota \Delta y} \right] \\ &= \left[\frac{\partial v}{\partial y} - \iota \frac{\partial u}{\partial y} \right]_{z_0=(x_0,y_0)} \end{aligned}$$

Since the two limits must be equal, that is, the respective real and imaginary parts must be equal. We get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Example-continuous but not differentiable

Example. $f(z) = \bar{z}$ is continuous at $z = 0$ but not differentiable.

Analytic Functions

Even though the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called Analytic Functions.

Analyticity at a point

A complex function is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighbourhood of z_0.

A function f is said to be analytic in a domain D if it is analytic at every point in D. Such a function is called holomorphic or regular.

Analytic Functions

- Analyticity at a point is not the same as differentiability at a point.
- Analyticity at a point is a neighbourhood property; in other words, analyticity is a property defined on an open set.

Example. $|z|^2$ is not analytic.

Example. $|z|^2$ is not analytic even when it is differentiable at $z = 0$.

Entire Functions

Entire Functions

A function that is analytic at every point z in the complex plane is said to be an Entire Function.

Theorem

1. *A polynomial function* $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, *where* n *is a nonnegative integer, is an entire function.*
2. *A rational function* $f(z) = \dfrac{p(z)}{q(z)}$, *where* p *and* q *are polynomial functions, is analytic in any domain* D *that contains no point* z_0 *for which* $q(z_0) = 0$.

Necessary Condition for Analyticity

If a function $f(z) = u(x, y) + \iota v(x, y)$ is differentiable at a point z, then the functions u and v must satisfy a pair of equations that relate their first-order partial derivatives.

Theorem (Cauchy-Riemann Equations)

Suppose $f(z) = u(x, y) + \iota v(x, y)$ is differentiable at a point $z = x + \iota y$, then at z the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann Equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \textit{and}\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The theorem states that C-R equations hold at z as a *necessary* consequence of f being differentiable at z, we cannot use the theorem to help us to determine where f is differentiable. *But* it is important to realize that the theorem tells us where a function f does not possess derivative.
If the C-R equations are *not* satisfied at a point z, then f cannot be differentiable at z.

Example not differentiable at any z

$f(z) = x + 4\iota y$ is not differentiable at any point z.

Criterion for Non-analyticity

Criterion for Non-analyticity

If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x, y) + \iota v(x, y)$ cannot be analytic in D.

Example - z^2 is analytic

Example

Example. For the function $f(z) = \dfrac{x}{x^2+y^2} - \iota \dfrac{y}{x^2+y^2}$, the real functions $u(x,y) = \dfrac{x}{x^2+y^2}$ and $v(x,y) = -\dfrac{y}{x^2+y^2}$ are continuous except at the point where $x^2+y^2=0$, that is, at $z=0$.

Example- CR equations satisfied but not differentiable

Example. The function

$$f(x) = \begin{cases} \dfrac{x^3(1+\iota) - y^3(1-\iota)}{x^2+y^2} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

satisfies the CR equations at $z = 0$ but $f'(0)$ does not exist.

Sufficient Condition for Analyticity

- By themselves, the Cauchy-Riemann equations do not ensure analyticity of a function $f(z) = u(x, y) + \iota v(x, y)$ at a point $z = x + \iota y$.
- It is possible for the C-R equations to be satisfied at z and yet $f(z)$ may not be differentiable at z, or $f(z)$ may be differentiable at z but nowhere else. In either case, f is not analytic at z.
- However, when we add the condition of continuity to u and v and to the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, it can be shown that the C-R equations are not only necessary but also sufficient to guarantee analyticity of $f(z) = u(x, y) + \iota v(x, y)$ at z.

Sufficient Condition for Analyticity

Theorem (Sufficient Condition for Analyticity)

Suppose the real function $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the C-R equations at all points of D, then the complex function $f(z) = u(x, y) + \iota v(x, y)$ is analytic in D and

$$f'(z) = u_x + \iota v_x = v_y - \iota v_x$$

Example

Example. Using C-R equations, verify the analyticity of
$(1).f(z) = |z|^2,\ (2).f(z) = \bar{z}.$

Exponential and Logarithmic Functions

- If x is a fixed positive real number, then there is a *single* solution to the equation $x = e^y$, namely the value $y = \log_e x$.
- However, when z is a fixed nonzero complex number then there are *infinitely* many solutions to the equation $z = e^w$. Therefore, the complex logarithmic function is a "multiple valued function".
- The principal value of the complex logarithm will be defined to be a single-valued function that assigns to the complex input z which is the inverse function of the exponential function e^z defined on a suitably restricted domain of the complex plane.

Example

Take $a > 0$ and not equal to 1. Then the function defined as

$$f : \mathbb{R} \to \mathbb{R}$$

given by

$$f(x) = a^x$$

is called an exponential function with base a.

The number e

We try to calculate the derivative of the exponential function $f(x) = a^x$.

$$\begin{aligned}
\frac{d}{dx}[f(x)] &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
\frac{d}{dx}[a^x] &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - 1}{h}
\end{aligned}$$

Now, $\lim\limits_{h\to 0} \dfrac{a^h - 1}{h}$ is a contant depending on the value of the base a. It can be proved that there is a unique value of a, such that the limit is equal to 1. This very special value of a is e. So,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

The number e as a limit

The expression

$$\lim_{h\to 0}\frac{e^h-1}{h}=1$$

means that for very small values of h

$$\begin{aligned} & e^h-1 \text{ is approximately } h \\ \iff\ & e^h \text{ is approximately } h+1 \\ \iff\ & e \text{ is approximately } (1+h)^{1/h} \end{aligned}$$

So,

$$e=\lim_{h\to 0}(1+h)^{1/h}=2.71828...$$

Or if we say that $t=1/h$, then

$$e=\lim_{t\to\infty}(1+1/t)^t=2.71828...$$

Complex Exponential Function

Complex Exponential Function

The function e^z defined by

$$e^z = e^x \cos y + \iota e^x \sin y$$

is called the Complex Exponential Function.

When z is real, the function agrees with the real exponential function.

Analyticity of e^z

Analyticity of e^z

The exponential function e^z is entire and its derivative is given by :

$$\frac{d}{dz}e^z = e^z$$

.

Note that the real and imaginary parts of the complex exponential function $u = e^x \cos y$ and $v = e^x \sin y$ are continuous real functions and have continuous first-order partial derivatives for all (x, y). In addition, the Cauchy-Riemann equations in u and v are easily verified:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

Therefore, the complex exponential function is an entire function and it's derivative is given by

$$\frac{d}{dz}e^z = \frac{\partial u}{\partial x} + \iota\frac{\partial v}{\partial x} = e^x \cos y + \iota e^x \sin y = e^z$$

Complex Trigonometric Functions

The formulas for real sine and cosine functions can be used to define complex sine and cosine functions by replacing the real variable x with the complex variable z.

Complex Sine and Cosine Functions

The complex sine and cosine functions are defined by :

$$\cos z = \frac{e^{\iota z} + e^{-\iota z}}{2} \text{ and } \sin z = \frac{e^{\iota z} - e^{-\iota z}}{2\iota}$$

Modulus of Complex Sine Function

$\sin z = \sin(x + \iota y) = \sin x \cosh y + \iota \cos x \sinh y$
Thus,

$$\begin{aligned} |\sin z| &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \\ &= \sqrt{\sin^2 x + \sinh^2 y} \end{aligned} \tag{2}$$

It may be recalled that the real hyperbolic function $\sinh x$ is unbounded on the real line. The expression in eq (2) can be made arbitrarily large by choosing y to be arbitrarily large. Thus, the complex sine function is not bounded on the complex plane, i.e., there does not exist a real constant M so that $|\sin z| < M$ for all $z \in \mathbb{C}$, which of course is different from the situation for the real sine function for which $|\sin x| \leq 1$ for all real x.

Analyticity of $\sin z$

Let $z = x + \iota y$. Then $\sin z = \sin x \cosh y + \iota \cos x \sinh y = u(x,y) + \iota v(x,y)$
Therefore, $u_x = \sin x \cosh y,\ v_y = \cos x \sinh y$

$$u_x = \cos x \cosh y = v_y,\ u_y = \sin x \sinh y = -v_x$$

Since, the CR equations are satisfied for all (x,y) and the frst order partial derivatives of $u(x,y), v(x,y)$ are continuous everywhere, the given function is analytic for all z in the finite $z-$plane. We obtain

$$\frac{d}{dz}\sin z = u_x + \iota v_x = \cos x \cosh y - \iota \sin x \sinh y = \cos z$$

Analyticity of $\cos z$

Let $z = x + \iota y$. Then $\cos z = \cos x \cosh y - \iota \sin x \sinh y = u(x, y) + \iota v(x, y)$
Therefore, $u_x = \cos x \cosh y$, $v_y = -\sin x \sinh y$

$$u_x = -\sin x \cosh y = v_y, \; u_y = \cos x \sinh y = -v_x$$

Since, the CR equations are satisfied for all (x, y) and the frst order partial derivatives of $u(x, y), v(x, y)$ are continuous everywhere, the given function is analytic for all z in the finite $z-$plane. We obtain

$$\frac{d}{dz} \cos z = u_x + \iota v_x = -\sin x \cosh y - \iota \cos x \sinh y = -\sin z$$

Harmonic Functions

It is known that when a complex function $f(z) = u(x, y) + \imath v(x, y)$ is analytic at a point z, then all the derivatives of f : $f'(z), f''(z), f'''(z), \cdots$ are also analytic at z. As a consequence of this fact, we can conclude that all partial derivatives of the real functions $u(x, y)$ and $v(x, y)$ are continuous at z. From the continuity of partial derivatives we then know that the second-order mixed partial derivatives are equal. This fact, together with the C-R equations, demonstrates that there is a connection between the real and imaginary parts of an analytic function $f(z)$ and the second-order partial differential equation.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

This equation is known as Laplace's equation in two variables.

Definition (Harmonic Function)

A real-valued function $\phi(x, y)$ of two variables x and y that has continuous first and second order partial derivatives in a domain D and satisfies Laplace's equation is said to be be harmonic in D.

Theorem (Harmonic Functions)

Suppose the complex function $f(z) = u(x, y) + \imath v(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

Example - Harmonic Function

Example. The function $f(z) = z^2 = x^2 - y^2 + 2xy\iota$ is entire. The functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are necessarily harmonic in any domain D of the complex plane.

Harmonic Conjugate Functions

If a function $f(z) = u(x, y) + \iota v(x, y)$ is analytic in a domain D, then its real and imaginary parts u and v are necessarily harmonic in D.
Now suppose $u(x, y)$ is a given real function that is known to be harmonic in D. If it is possible to find another real harmonic function $v(x, y)$ so that u and v satisfy the C-R equations throughout the domain D, then the function $v(x, y)$ is called a harmonic conjugate of $u(x, y)$.
By combining the functions as $u(x, y) + \iota v(x, y)$ we obtain a function that is analytic in D.

Applications of Harmonic Functions

A wide variety of problems in engineering and physics involve harmonic functions, which are real or the imaginary part of an analytic function. The standard applications are

- two dimensional steady state temperatures
- electrostatics
- fluid flows
- complex potentials

And a more recent one is to the robotics.

Applications to Flow Problems

The complex potential $w(z)$ canbe taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In the heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and vice versa.

Example. Find harmonic conjugate of $u = x^3 - 3xy^2 - 5y$

Some Standard Transformations

Here, it will be shown that every nonconstant complex linear mapping can be described as a composition of three basic types of motions :

1. a translation
2. a rotation
3. a magnification
4. a reciprocal (or an inversion)

Translation

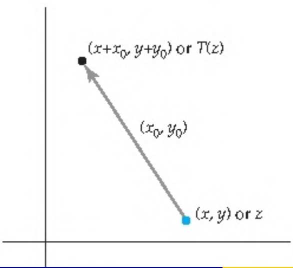
Definition (Translation)

A complex linear function

$$T(z) = z + b,\ b \neq 0 \tag{3}$$

is called a **translation**. If we set $z = x + \iota y$ and $b = x_0 + \iota y_0$ in (3), then we obtain

$$T(x + \iota y) = x + x_0 + \iota(y + y_0)$$



From image, we see that if we plot (x, y) and $(x + x_0, y + y_0)$ in the same copy of the complex plane, then the vector originating at (x, y) and terminating at $(x + x_0, y + y_0)$ is (x_0, y_0), which is the vector representation of the complex number b, the mapping $T(z)$ is also called the *translation by b*.

Example. Image of a Square under translation

Find the image S' of the square S with vertices at $1+\iota,\ 2+\iota,\ 2+2\iota,\ 1+2\iota$ under the linear mapping $T(z) = z + 2 - \iota$.

Rotation

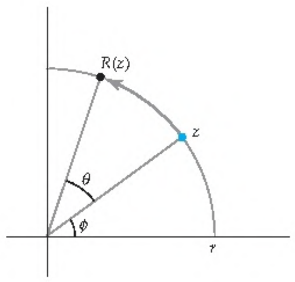
Definition (Rotation)

A complex linear function

$$R(z) = az, \ |a| = 1 \tag{4}$$

is called a **rotation**.

The condition $|a| = 1$ is not a major requirement. If α is any nonzero complex number, then $a = \alpha/|\alpha|$ is a complex number for which $|a| = 1$ and in that case $R(z) = \dfrac{\alpha}{|\alpha|}z$ is a rotation.



If we write $a = e^{\iota\theta}$ and $z = re^{\iota\phi}$ in (4), we obtain $R(z) = re^{\iota(\theta+\phi)}$. Modulus of $R(z)$ is same as z i.e. r. From the image, it is clear that both the points lie on a circle centred at 0 and radius r. Clearly, the mapping $R(z) = az$ can be visualized in a single copy of the complex plane as the process of *rotating* the point z *counterclockwise* or *clockwise* through an angle of θ radians about the origin to the point $R(z)$ if $Arg(a) > 0$ or $Arg(z) < 0$ respectively. Thus, $\theta = Arg(a)$ is called the angle or rotation of R.

Example. Image of a Line under Rotation

Find the image of the real axis $y = 0$ under the linear mapping

$$R(z) = \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\imath\right) z$$

Magnification

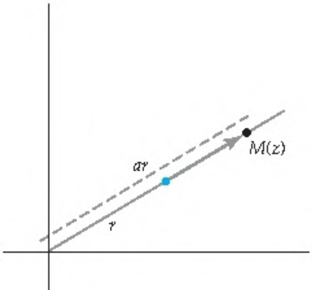
Definition (Magnification)

A complex linear function

$$M(z) = az, a > 0 \tag{5}$$

is called a **magnification**. If $z = x + \iota y$, then $M(z) = az = ax + \iota ay$ and so the image of the point (x, y) is the point (ax, ay). Using the exponential form, we can express eq (5) as

$$M(z) = a(re^{\iota\theta}) = (ar)e^{\iota\theta} \tag{6}$$



Assuming $a > 1$, we know that both z and $M(z)$ have same argument θ but different moduli. Plotting z and $M(z)$ in the same copy of complex plane, then $M(z)$ is the unique point on the ray emanating from 0 and containing z whose distance from 0 is ar. The point $M(z)$ is a times *farther* or *closer* from the origin than z depending on whether $a > 1$ or $0 < a < 1$ respectively. a is called the *magnification factor of M*.

Example. Image of a Circle under Magnification

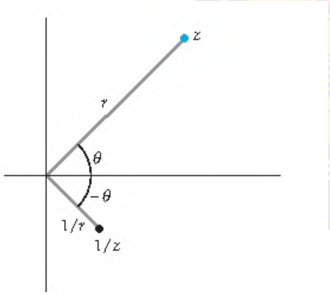
Find the image of a circle C given by $z = 2$ under the mapping $M(z) = 3z$.

Reciprocal (or Inversion)

Definition (Reciprocal)

The function $1/z$, whose domain is the set of all nonzero complex numbers, is called the **reciprocal function**. Given $z \neq 0$, if we set $z = re^{\iota\theta}$, then we obtain

$$w = \frac{1}{z} = \frac{1}{re^{\iota\theta}} = \frac{1}{r}e^{-\iota\theta} \tag{7}$$



Thus, the modulus of w is the reciprocal of the modulus of z and argument of w is negative of argument of z. Hence, the reciprocal function maps a point in the z-plane with the polar co-ordinates (r, θ) onto a point in the w-plane with the polar co-ordinates $(1/r, -\theta)$. it is clear from the image, that the reciprocal function is a composition of inversion in the unit circle followed by reflection across the real axis.

Inversion in the Unit Circle

Definition (Inversion in the Unit Circle)

The function

$$g(z) = \frac{1}{z} e^{\iota\theta} \tag{8}$$

whose domain is the set of all nonzero complex numbers, is called inversion in the unit circle.

We will describe this mapping by considering separately the images of the points

- *on* the unit circle
- *outside* the unit circle
- *inside* the unit circle

Point on the Unit Circle

Consider a point *on* the unit circle. Then

$$z = 1.e^{\iota\theta}$$

Thus from (8), it is clear that

$$g(z) = \frac{1}{1}e^{\iota\theta} = z$$

Therefore, each point on the unit circle is mapped onto itself by g.

Point not on the Unit Circle

If, on the other hand, z is a nonzero complex number that does not lie on the unit circle, then we can write z as $z = re^{\iota\theta}$ with $r \neq 1$.

Case I When $r > 1$, (i.e. when z is outside of the unit circle), we have

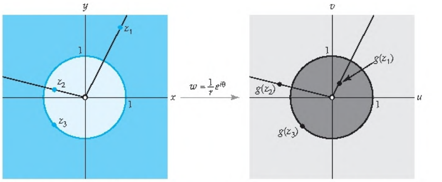
$$|g(z)| = \left|\frac{1}{r}e^{\iota\theta}\right| = \frac{1}{r} < 1$$

So the image under g of a point z outside the unit circle is a point inside the unit circle.

Case II If $r < 1$, (i.e. when z is inside the unit circle), then

$$|g(z)| = \frac{1}{r} > 1$$

and we conclude that image under g of a point z inside the unit circle is a point outside the unit circle.



(a) Points z_1, z_2, and z_3 in the z-plane (b) The images of z_1, z_2, and z_3 in the w-plane

Example. Image of Semicircle $|z| = 2,\ 0 \leq \arg(z) \leq \pi$ under $w = 1/z$

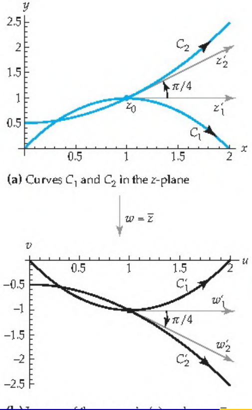
Example. Image of $x = 1$ under $w = 1/z$

Conformal Transformations

In previous sections, we saw nonconstant linear mapping acts by rotating, magnifying and translating points in the complex plane. As a result, the angle between any two intersecting arcs in the z-plane is equal to the angle between the images of the arcs in the w-plane under a linear mapping. Complex mappings that have this angle-preserving property are called Conformal Transformations.

Example. NonConformal Transformations

Consider two smooth curves C_1 and C_2 given by $z_1(t) = t + (2t - t^2)\iota$ and $z_2(t) = 1 + \frac{1}{2}(t^2 + 1)\iota$, $0 \le t \le 2$, respectively. These curves intersect at $z_0 = z_1(1) = z_2(1) = 1 + \iota$. Under the transformation $w = \bar{z}$, the images C_1' and C_2' in the w-plane are given by $w_1(t) = t - (2t - t^2)\iota$ and $w_2(t) = 1 - \frac{1}{2}(t^2 + 1)\iota$ and intersect at the point $w_0 = f(z_0) = 1 - \iota$.



- Tangent vectors in z-plane are $z_1' = 1,\ z_2' = 1 + \iota$. Moreover, the angle between C_1 and C_2 at z_0 is $\theta = \pi/4$.
- Tangent vectors in w-plane are $w_1' = 1,\ w_2' = 1 - \iota$. Also, the angle between C_1' and C_2' at w_0 is $\phi = \pi/4$.
- Thus, the angles are equal in magnitude.
- However, the rotation in z-plane is *counterclockwise* whereas in w-plane it is *clockwise*, thus θ and

Conformal Transformations

Definition

Let $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. Then we say that $w = f(z)$ is conformal at z_0 if for every pair of smooth oriented curves C_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C_1' and C_2' at $f(z_0)$ in both magnitude and sense.

Bilinear Transformations

In many applications that involve boundary-value problems associated with Laplace's equation, it is necessary to find a conformal mapping that maps a disk onto the half-plane $v \geq 0$. Such a mapping would have to map the circular boundary of the disk to the boundary line of the half-plane. An important class of elementary conformal mappings that map circles to lines (and vice versa) are the linear fractional transformations or Mobius transformations or bilinear transformations.

Bilinear Transformations

Definition (Bilinear Transformation)

If a, b, c, d are complex constants with $ad - bc \neq 0$, then the complex function defined by

$$T(z) = \frac{az + b}{cz + d} \tag{9}$$

is called a Bilinear transformation.

If $c = 0$, then the transformation given by (9) is a linear mapping and so a linear mapping is a special case of a bilinear transformation.
If $c \neq 0$, then we can write

$$T(z) = \frac{az + b}{cz + d} = \frac{bc - ad}{c} \frac{1}{cz + d} + \frac{a}{c} \tag{10}$$

It is clear from (10) is a combination of all the basic transformations studied earlier.

Bilinear Transformations

The domain of a bilinear transformation T given by (9) is the set of all complex z such that $z \neq -d/c$. Furthermore, since

$$T'(z) = \frac{ad - bc}{(cz + d)^2}$$

where $ad - bc \neq 0$, linear transformations are conformal on their domains. Also that T is a one-to-one function on its domain.

Cross Ratio

In applications we often need to find a conformal mapping from a domain D that is bounded by circles onto a domain D' that is bounded by lines. Bilinear transformations are particularly well-suited for such applications. However, in order to use them, we must determine a general method to construct a bilinear transformation $w = T(z)$, which maps three given distinct points z_1, z_2, z_3 on the boundary of D to three given distict points w_1, w_2, w_3 on the boundary of D'. This is accomplished using the cross-ratio.

Cross-Ratio

Definition

The cross-ratio of the complex numbers z, z_1, z_2, z_3 is the complex number

$$\frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} \tag{11}$$

When computing a cross-ratio, we must be careful with the order of the complex numbers.
The cross-ratio of ∞, z_1, z_2, z_3 is defined as the limit as given below:

$$\lim_{z\to\infty}\frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} \tag{12}$$

Cross-Ratios and Bilinear Transformations

Theorem

If $w = T(z)$ is a bilinear transformation that maps the distint points z_1, z_2, z_3 onto the distinct points w_1, w_2, w_3 respectively, then

$$\frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3}\frac{w_2 - w_3}{w_2 - w_1} \tag{13}$$

for all z.

Example. Bilinear Transformation

Construct a bilinear transformation that maps the points $1, \iota, -1$ on the unit circle $|z| = 1$ onto the points $-1, 0, 1$ on the real axis. Determine the image of the interior of $|z| < 1$ under this transformation.

Example. Bilinear Transformation

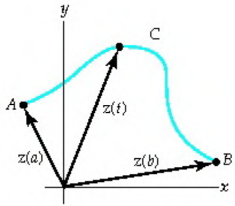
Construct a bilinear transformation that maps the points $-\iota, 1, \infty$ on the line $y = x - 1$ onto the points $1, \iota, -1$ on the real axis.

Curves Revisited

Suppose the continuous real-valued functions $x = x(t), y = y(t), a \leq t \leq b$ are parametric equations of a curve C in the complex plane. If we use these equations as the real and imaginary parts in $z = x + \iota y$ then we can describe the points z on C by means of a complex-valued function of a real variable t called a **parametrization** of C:

$$z(t) = x(t) + \iota y(t), a \leq t \leq b \tag{14}$$

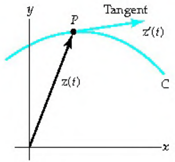
For example, the parametric equations $x = \cos t, y = \sin t,\ 0 \leq t \leq 2\pi$, describe a unit circle centered at origin. A parametrization of this circle is $z(t) = \cos t + \iota \sin t$ or $z(t) = e^{\iota t},\ 0 \leq t \leq 2\pi$.

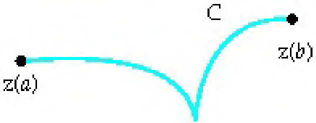


The point $z(a) = x(a) + \iota y(a)$ or $A = (x(a), y(a))$ is called the **initial point** of C and $z(b) = x(b) + \iota y(b)$ or $B = (x(b), y(b))$ is called the **terminal point** of C. $z(a)$ and $Z(b)$ are interpreted as position vectors. As t varies from $t = a$ to $t = b$ we can envision the curve C being traced out by the moving arrowhead of $z(t)$.

Contours

Supppose the derivative of (14) is $z'(t) = x'(t) + \imath y'(t)$. We say a curve C in the complex plane is smooth if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$, as shown in Figure (a), the vector $z'(t)$ is tangent to C at P. Thus, a smooth curve has continuously turning tangent, i.e. a smooth curve can have no sharp corners or cusps.





- A piecewise smooth curve C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \cdots, C_n$ are joined together.
- A curve C is said to be simple if $z(t_1) \neq z(t_2), t_1 \neq t_2$, except possibly for $t = a$ and $t = b$.
- C is called a closed curve if $z(a) = z(b)$.
- C is called a simple closed curve if $z(t_1) \neq z(t_2), t_1 \neq t_2$ and $z(a) = z(b)$.
- In complex analysis, a piecewise smooth curve C is called a contour or path

Complex Integral

Definition (Complex Integral)

An integral of a function f of a complex variable z that is defined on a contour C is denoted by $\int_C f(z)dz$ and is called a **complex integral**.

Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by the parametrization $z(t) = x(t) + \imath y(t),\ a \leq t \leq b$ then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{15}$$

Example. Evaluating a Contour Integral

Example. Evaluate $\int_C \bar{z} dz$, where C is given by $x = 3t, y = t^2, \ -1 \leq t \leq 4$.
Solution. The parametrization of C is $z(t) = 3t + \iota t^2$.
Thus, we have $f(z) = \bar{z} = 3t - \iota t^2$.
Also, $z'(t) = 3 + 2\iota t$ and thus the integral becomes,

$$\begin{aligned} \int_C \bar{z} dz = \int_{-1}^{4} (3t - \iota t^2)(3 + 2\iota t) dt &= \int_{-1}^{4} [2t^3 + 9t + 3t^2 \iota] dt \\ &= \left(\frac{1}{2} t^4 + \frac{9}{2} t^2 \right) \Big|_{-1}^{4} + \iota t^3 \Big|_{-1}^{4} \\ &= 195 + 65\iota \end{aligned} \tag{16}$$

Properties of Contour Integrals

Theorem (Properties of Contour Integrals)

Suppose the functions f and g are continous in domain D and C is smooth curve lying entirely in D. Then

- $\displaystyle\int_c kf(z)dz = k\int_C f(z)dz$, *$k$ is a complex constant.*
- $\displaystyle\int_c [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$
- $\displaystyle\int_c f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, *where C consists of the smooth curves C_1 and C_2 joined end to end.*
- $\displaystyle\int_{-c} f(z)dz = -\int_C f(z)dz$, *where $-C$ denotes the curve having opposite orientation of C.*

Example. Contour Integration

Example. Evaluate $I = \int_C (x^2 + \iota y^2) dz$, where C is the contour shown in figure.

Simply and Multiply Connected Domains

Definition

A domain D is simply connected if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D.

A simply connected domain has no "holes" in it. The entire complex plane is an example of a simply connected domain; the annulus defined by $1 < |z| < 2$ is not simply connected.

Definition

A domain that is not simply connected is called a multiply connected domain; that is, a multiply connected domain has "holes" in it.

A domain with one "hole" is called doubly connected, a domain with two holes is called triply connected and so on.
The open disc defined by $|z| < 2$ is a simply connected domain; the open circular annulus defined by $1 < |z| < 2$ is a doubly connected domain.

Cauchy's Theorem

Theorem (Cauchy's Theorem)

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\int_C f(z)dz = 0$.

Cauchy-Goursat Theorem

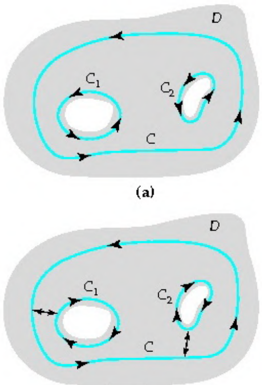
In 1883, the French mathematician Edouard Goursat proved that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem. The resulting modified version of Cauchy's theorem is known today as the Cauchy-Goursat Theorem.

Theorem (Cauchy-Goursat Theorem)

Suppose that a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D, $\oint_C f(z)dz = 0.$

Observation

If C, C_1, C_2 are simple closed contours as shown in figure (a) and if f is analytic on each of the three contours as well as at each point interior to C but exterior to both C_1 and C_2, then by introducing crosscuts between C_1 and C and between C_2 and C, as illustrated in figure (b), it follows from the theorem that



$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$

and so

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

Consequences of Cauchy-Goursat Theorem

The most significant consequences of Cauchy-Goursat Theorem are:

- The value of an analytic function f at any point Z_0 in a simply connected domain can be represented by a contour integral.
- An analytic function f in a simply connected domain possesses derivatives of all orders.

Cauchy's Integral Formula

Theorem (Cauchy's Integral Formula)

If $f(z)$ is analytic within and on a closed curve and if z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi\iota} \oint_C \frac{f(z)}{z - z_0} dz \tag{17}$$

COROLLARY

Differentiating both the sides of (17) w.r.t. z_0, we get

$$f'(z_0) = \frac{1}{2\pi\iota}\oint_C \frac{\partial}{\partial z_0}\left[\frac{f(z)}{z-z_0}\right]dz = \frac{1}{2\pi\iota}\oint_C \frac{f(z)}{(z-z_0)^2}dz \quad (18)$$

$$Similarly\ f''(z_0) = \frac{2!}{2\pi\iota}\oint_C \frac{f(z)}{(z-z_0)^3}dz \quad (19)$$

$$In\ general\ f^{(n)}(z_0) = \frac{n!}{2\pi\iota}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \quad (20)$$

- Thus, it follows from the results (17) to (20) that if a function $f(z)$ is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C.
- Incidently, a remarkable fact is established that an analytic function possesses derivatives of all orders and these are themselves all analytic.

Example

Example. Evaluate $I = \oint_C (z-a)^{-1} dz$, where C is a simple closed curve and the point $z = a$ is (i) outside C, (ii) inside C.

Example

Example. Evaluate $\oint_C \frac{dz}{(z-a)^n} dz,\ n = 2, 3, 4, \cdots$ where C is a closed curve containing the point $z = a$.

Example

Example. Use Cauchy's integral formula to evaluate $I = \oint_C \frac{3z^2 + 7z + 1}{z+1} dz$, where C is $|z| = \frac{1}{2}$.

Example

Example. Use Cauchy's integral formula to evaluate $I = \oint_C \frac{2z+1}{z^2+z} dz$, where C is $|z| = \frac{1}{2}$.

Example

Example. Use Cauchy's integral formula to evaluate $I = \oint_C \frac{e^z}{(z+1)^2} dz$, where C is $|z-1| = 3$.

Example

Example. Use Cauchy's integral formula to evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2 - z - 2}{z - a} dz$ and C is $|z| = 2.5$.

Taylor's Series

Theorem (Taylor's Series)

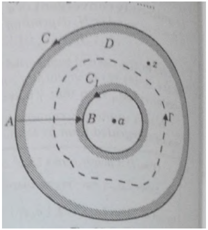
If $f(z)$ is analytic inside a circle C with centre at z_0, then for any z inside C,

$$f(z) = f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\cdots \quad (21)$$

Laurent's Series

Theorem (Laurent's Series)

If $f(z)$ is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 $(r > r_1)$ and with centre at z_0, then for all z in R,



$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_{-1}(z - z_0)^{-1} + a_{-2}(z - z_0)^{-2} + \cdots \quad (22)$$

where $a_n = \dfrac{1}{2\pi\iota}\displaystyle\int_\Gamma \frac{f(t)}{(t - z_0)^{n+1}} dt$

Γ *being any curve in R, encircling C_1*

Example. Taylor Series

Example. Expand the function $f(z) = \dfrac{1}{z}$ about $z = 2$ in Taylor's series. Obtain its radius of convergence.

Example. Taylor Series

Example. Obtain the Taylor series expansion of $f(z) = \dfrac{1}{z^2 + (1 + 2\iota)z + 2\iota}$ about $z = 0$. Also find its radius.

Zeros of an Analytic Function

Definition (Zeros of an Analytic Function)

A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$. If $f(z)$ is analytic in the neighbourhood of the point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n + \cdots, \ where\ a_n = \frac{f^{(n)}(a)}{n!}$$

If $a_0 = a_1 = \cdots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a zero of order m at $z = a$. When $m = 1$, the zero is said to be simple.

Singularities of an Analytic Function

Definition (Singularities of an Analytic Function)

A singular point of a function is the point at which the function ceases to be analytic.

Definition (Isolated Singularity)

If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood (i.e. there exists a circle with centre a which has no other singularity), then $z = a$ is caled an isolated singularity. In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots \quad (23)$$

Singularities of an Analytic Function

Definition (Removable Singularity)

If all the negative powers of $z-a$ in (23) are zero, then $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$.

Here, the singularity can be removed by defining $f(z)$ at $z=a$ in such a way that it becomes analytic at $z=a$. Such a singularity is called removable singularity.

Thus is $\lim_{z\to a} f(z)$ exists finitely, then $z=a$ is a removable singularity.

Definition (Poles)

If all the negative powers of $z-a$ in (23) after the n^{th} are missing, then the singularity at $z=a$ is called a pole of order n. A pole of first order is called a simple pole.

Definition (Essential Singularity)

If the number of negative powers of $z-a$ is (23) is infinite, then $z=a$ is called an essential singularity. In this case, $\lim_{z\to a} f(z)$ does not exist.

Residues and Residue Theorem

If a complex function f has an isolated singularity at a point z_0, then f has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

which converges for all z near z_0.

Definition (Residue)

The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series given above is called the residue of the function f at the isolated singularity z_0. We shall use the notation

$$a_{-1} = Res(f(z), z_0)$$

to denote the residue of f at z_0.

Example

Consider the function $f(z) = \dfrac{1}{(z-1)^2(z-3)}$.

Residue at Pole or order 1 or n

Definition (Residue at a Simple Pole)

If f has a simple pole at $z = z_0$, then

$$Res(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{24}$$

Definition (Residue at a Pole of order n)

If f has a pole of order n at $z = z_0$, then

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{25}$$

Example. Residue at a pole

Example. Consider the function $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$.

Cauchy's Residue Theorem

Theorem (Cauchy's Residue Theorem)

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of isolated singular points $z_1, z_2, \cdots, z_n$ within C, then

$$\oint_C f(z)dz = 2\pi\iota \sum_{k=1}^{n} Res(f(z), z_k) \tag{26}$$

Example.

Example. Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}dz$, where
(a) the contour C is the rectangle defined by $x = 0, x = 4, y = -1, y = 1$
(b) the contour C is the circle $|z| = 2$.

Example.

Example. Evaluate $\oint_C \frac{2z+6}{z^2+4}dz$, where the contour C is the circle $|z-\iota|=2$.

Example.

Example. Evaluate

- $\displaystyle\oint_C \frac{e^z}{z^4+5z^3}dz$, where $C : |z| = 2$.
- $\displaystyle\oint_C \tan z dz$, where $C : |z| = 2$.
- $\displaystyle\oint_C \frac{1}{z\sin z}dz$, where $C : |z-2\imath| = 1$.

Evaluation of Real Integrals

The basic idea here is to convert a real trigonometric integral of form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$ into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin.

Put $z = e^{\iota\theta}$. Then $dz = \iota e^{\iota\theta}d\theta$, $\cos\theta = \dfrac{e^{\iota\theta} + e^{-\iota\theta}}{2}$, $\sin\theta = \dfrac{e^{\iota\theta} - e^{-\iota\theta}}{2\iota}$. Thus, we may write $d\theta = \dfrac{dz}{\iota z}$, $\cos\theta = \dfrac{1}{2}(z + z^{-1})$, $\sin\theta = \dfrac{1}{2\iota}(z - z^{-1})$. Thus, the given integral can be written as

$$\oint_C F\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2\iota}(z - z^{-1})\right) \frac{dz}{\iota z}$$

where C is the unit circle $|z| = 1$.

Example

Example. Evaluate

- $\displaystyle\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\,d\theta$
- $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\,d\theta$
- $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$

THANK YOU!!