Matrices

Introduction

Matrices - Introduction

Matrix algebra has at least two advantages:

•Reduces complicated systems of equations to simple expressions

•Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

Properties:

•A specified number of rows and a specified number of columns

•Two numbers (rows x columns) describe the dimensions or size of the matrix.

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

MATRICES - INTRODUCTION

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{1n} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

Where *i* goes from 1 to m and *j* goes from 1 to *n*

Matrices - Introduction

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Matrices - Introduction

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $\mathbf{A}_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Matrices - Introduction

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for some or all $i = j$.

Matrices - Introduction

TYPES OF MATRICES

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all i,j

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TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

MATRICES – INTRODUCTION
TYPES OF MATRICES

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$
$a_{ij} = a$ for all $i = j$

Matrices

Matrix Operations

Matrices - Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \mathbf{B}$$

Matrices - Operations

Some properties of equality:

•If **A** = **B**, then **B** = **A** for all **A** and **B**

•If **A** = **B**, and **B** = **C**, then **A** = **C** for all **A**, **B** and **C**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If **A** = **B** then $a_{ij} = b_{ij}$

Matrices - Operations

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size.

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted.

Matrices - Operations

Commutative Law:

A + **B** = **B** + **A**

Associative Law:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A	**B**	**C**
2x3	2x3	2x3

MATRICES - OPERATIONS

A + **0** = **0** + **A** = **A**

A + (-**A**) = **0** (where –**A** is the matrix composed of $-a_{ij}$ as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Matrices - Operations

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Matrices - Operations

$$4\times\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}\times 4=\begin{bmatrix}12 & -4\\ 8 & 4\\ 8 & -12\\ 16 & 4\end{bmatrix}$$

Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

MATRICES - OPERATIONS

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

A	x **B**	=	**C**
(1x3)	(3x1)		(1x1)

Matrices - Operations

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible!

(6x2) (6x3)

Example

A x **B** = **C**

(2x3) (3x2) (2x2)

MATRICES - OPERATIONS

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

Matrices - Operations

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = **A**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Matrices - Operations

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI** = **IA** = **A**
2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)
3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)
4. (**A**+**B**)**C** = **AC** + **BC** - (second distributive law)

Caution!

1. **AB** not generally equal to **BA**, **BA** may not be conformable
2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**
3. If **AB** = **AC**, **B** not necessarily = **C**

Matrices - Operations

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

Matrices - Operations

If **AB** = **0**, neither **A** nor **B** necessarily = **0**

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices - Operations

TRANSPOSE OF A MATRIX

If :

$$\underset{2\text{x}3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T$$ For all *i* and *j*

Matrices - Operations

To transpose:

Interchange rows and columns

The dimensions of $\mathbf{A}^{\mathrm{T}}$ are the reverse of the dimensions of **A**

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2 x 3

$$A^T = {}_3A^{T\,2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

3 x 2

Matrices - Operations

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Matrices - Operations

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

Matrices - Operations

$(\mathbf{AB})^T = \mathbf{B}^T\,\mathbf{A}^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

Matrices - Operations

SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Matrices - Operations

When the original matrix is square, transposition does not affect the elements of the main diagonal

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The identity matrix, **I**, a diagonal matrix **D**, and a scalar matrix, **K**, are equal to their transpose since the diagonal is unaffected.

Matrices - Operations

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

$k = 7$ the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be $\mathbf{AB} = \mathbf{AC}$ while $\mathbf{B} \neq \mathbf{C}$

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

Matrices - Operations

Example:

$$A{=}_{2}A^{2} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices - Operations

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix

A matrix that does not have an inverse is called a singular matrix

Square matrices have inverses except when the determinant is zero

When the determinant of a matrix is zero the matrix is singular

Matrices - Operations

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or |**A**|

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

Matrices - Operations

If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

Then |**A**| =det **A** = a_{11}

If **A** is (n x n), its determinant may be defined in terms of order (*n*-1) or less.

Matrices - Operations

MINORS

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where i and j correspond to the deleted

row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

Matrices - Operations

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Matrices - Operations

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices - Operations

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Matrices - Operations

DETERMINANTS CONTINUED

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Matrices - Operations

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices - Operations

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Matrices - Operations

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Matrices - Operations

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Matrices - Operations

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Matrices - Operations

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If $$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is $$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

Matrices - Operations

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adj A = C^T$$

It can be shown that:

A(adj **A**) = (adj**A**) **A** = |**A**| **I**

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adj A = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Matrices - Operations

$$A(adjA)=\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10I$$

$$(adjA)A=\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10I$$

Matrices - Operations

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Matrices - Operations

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices - Operations

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

$$|\mathbf{A}| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2$$

The elements of the cofactor matrix are

$$c_{11} = +(-1), \quad c_{12} = -(-2), \quad c_{13} = +(3),$$
$$c_{21} = -(-1), \quad c_{22} = +(-4), \quad c_{23} = -(7),$$
$$c_{31} = +(-1), \quad c_{32} = -(-2), \quad c_{33} = +(5),$$

Matrices - Operations

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Matrices - Operations

The result can be checked using

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

Matrix Inversion

Simple 2 x 2 case

Simple 2 x 2 case

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Since it is known that

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simple 2 x 2 case

Multiplying gives

$$aw + by = 1$$

$$ax + bz = 0$$

$$cw + dy = 0$$

$$cx + dz = 1$$

It can simply be shown that

$$|A| = ad - bc$$

Simple 2 x 2 case

thus

$$y = \frac{1 - aw}{b}$$

$$y = \frac{-cw}{d}$$

$$\frac{1 - aw}{b} = \frac{-cw}{d}$$

$$w = \frac{d}{da - bc} = \frac{d}{|A|}$$

Simple 2 x 2 case

$$z = \frac{-ax}{b}$$

$$z = \frac{1-cx}{d}$$

$$\frac{-ax}{b} = \frac{1-cx}{d}$$

$$x = \frac{b}{-da+bc} = -\frac{b}{|A|}$$

Simple 2 x 2 case

$$w = \frac{1 - by}{a}$$

$$w = \frac{-dy}{c}$$

$$\frac{1 - by}{a} = \frac{-dy}{c}$$

$$y = \frac{c}{-ad + cb} = -\frac{c}{|A|}$$

Simple 2 x 2 case

$$x = \frac{-bz}{a}$$

$$x = \frac{1-dz}{c}$$

$$\frac{-bz}{a} = \frac{1-dz}{c}$$

$$z = \frac{a}{ad-bc} = \frac{a}{|A|}$$

Simple 2 x 2 case

So that for a 2 x 2 matrix the inverse can be constructed in a simple fashion as

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \frac{d}{|A|} & \frac{b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of main diagonal
- Change sign in elements off main diagonal
- Divide resulting matrix by the determinant

Simple 2 x 2 case

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Check inverse

A^{-1} **A**=**I**

$$-\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices and Linear Equations

Linear Equations

Linear Equations

Linear equations are common and important for survey problems

Matrices can be used to express these linear equations and aid in the computation of unknown values

Example

n equations in *n* unknowns, the a_{ij} are numerical coefficients, the b_i are constants and the x_j are unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Linear Equations

The equations may be expressed in the form

$$\mathbf{AX} = \mathbf{B}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n1} \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

n x n n x 1 n x 1

Number of unknowns = number of equations = n

Linear Equations

If the determinant is nonzero, the equation can be solved to produce n numerical values for x that satisfy all the simultaneous equations

To solve, premultiply both sides of the equation by $\mathbf{A}^{-1}$ which exists because $|\mathbf{A}| \neq \mathbf{0}$

$$\mathbf{A}^{-1}\,\mathbf{AX} = \mathbf{A}^{-1}\,\mathbf{B}$$

Now since

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

We get

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

So if the inverse of the coefficient matrix is found, the unknowns, **X** would be determined

Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linear Equations

When $\mathbf{A}^{-1}$ is computed the equation becomes

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore

$$x_1 = 2,$$
$$x_2 = -3,$$
$$x_3 = -7$$

Linear Equations

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2, \quad x_2 = -3, \quad x_3 = -7$$

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 - x_3 &= 3 \end{aligned}$$

$$\begin{aligned} 3\times(2) - (-3) + (-7) &= 2 \\ 2\times(2) + (-3) &= 1 \\ (2) + 2\times(-3) - (-7) &= 3 \end{aligned}$$

Eigenvalues and Eigenvectors

1 Eigen values and Eigenvectors

2 Diagonalization

3 Symmetric Matrices and Orthogonal Diagonalization

4 Application of Eigenvalues and Eigenvectors

5 Principal Component Analysis

1 Eigenvalues and Eigenvectors

- Eigenvalue problem (one of the most important problems in the linear algebra):

If A is an $n \times n$ matrix, do there exist nonzero vectors $\mathbf{x}$ in R^n such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

(The term eigenvalue is from the German word *Eigenwert*, meaning "proper value")

- Eigenvalue and Eigenvector :

A: an $n \times n$ matrix

λ: a scalar (could be **zero**)

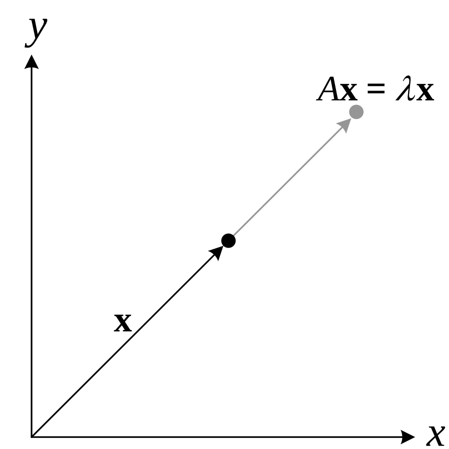
$\mathbf{x}$: a **nonzero** vector in R^n

Eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}$$

Eigenvector

※ Geometric Interpretation



- Ex 1: Verifying eigenvalues and eigenvectors

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \quad \mathbf{x}_1=\begin{bmatrix}1\\ 0\end{bmatrix} \quad \mathbf{x}_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$A\mathbf{x}_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=\overset{\substack{\text{Eigenvalue}\\ \downarrow}}{2}\underset{\substack{\uparrow\\ \text{Eigenvector}}}{\begin{bmatrix}1\\ 0\end{bmatrix}}=2\mathbf{x}_1$$

$$A\mathbf{x}_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=\overset{\substack{\text{Eigenvalue}\\ \downarrow}}{-1}\underset{\substack{\uparrow\\ \text{Eigenvector}}}{\begin{bmatrix}0\\ 1\end{bmatrix}}=(-1)\mathbf{x}_2$$

- Thm. 1: The eigenspace corresponding to λ of matrix A

If A is an $n\times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ **together with the zero vector** is a subspace of R^n. This subspace is called the eigenspace of λ

Proof:

$\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to λ

(i.e., $A\mathbf{x}_1 = \lambda\mathbf{x}_1,\ \ A\mathbf{x}_2 = \lambda\mathbf{x}_2$)

(1) $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$

(i.e., $\mathbf{x}_1 + \mathbf{x}_2$ is also an eigenvector corresponding to λ)

(2) $A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\lambda\mathbf{x}_1) = \lambda(c\mathbf{x}_1)$

(i.e., $c\mathbf{x}_1$ is also an eigenvector corresponding to λ)

Since this set is closed under vector addition and scalar multiplication, this set is a subspace of R^n .

- Ex 3: Examples of eigenspaces on the *xy*-plane

For the matrix A as follows, the corresponding eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

For the eigenvalue $\lambda_1 = -1$, corresponding vectors are any vectors on the x-axis

$$A\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

※ Thus, the eigenspace corresponding to $\lambda = -1$ is the x-axis, which is a subspace of R^2

For the eigenvalue $\lambda_2 = 1$, corresponding vectors are any vectors on the y-axis

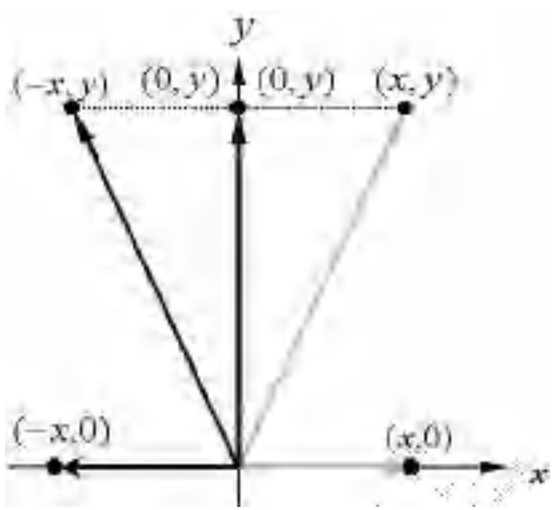
$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

※ Thus, the eigenspace corresponding to $\lambda = 1$ is the y-axis, which is a subspace of R^2

※ Geometrically speaking, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection to the y-axis, i.e., left multiplying A to $\mathbf{v}$ can transform $\mathbf{v}$ to another vector in the same vector space

$$A\mathbf{v} = A\begin{bmatrix} x \\ y \end{bmatrix} = A\left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ 0 \end{bmatrix} + A\begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$= -1\begin{bmatrix} x \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



- Thm. 2: Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$

  Let A be an $n\times n$ matrix.

  (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$

  (2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$

- Note: follwing the definition of the eigenvalue problem

  $A\mathbf{x} = \lambda\mathbf{x} \;\Rightarrow\; A\mathbf{x} = \lambda I\mathbf{x} \;\Rightarrow\; (\lambda I - A)\mathbf{x} = \mathbf{0}$ (homogeneous system)

  $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nonzero solutions for $\mathbf{x}$ iff $\det(\lambda I - A) = 0$

  (The above iff results comes from the equivalent conditions on Slide 4.101)

- Characteristic equation of A:

  $$\det(\lambda I - A) = 0$$

- Characteristic polynomial of $A \in M_{n\times n}$:

  $$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Ex 4: Finding eigenvalues and eigenvectors

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalue: $\lambda_1 = -1, \lambda_2 = -2$

$$(1)\ \lambda_1 = -1 \ \Rightarrow (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2)\ \lambda_2 = -2 \ \Rightarrow (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3s \\ s \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad s \neq 0$$

- Ex 5: Finding eigenvalues and eigenvectors

  Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of $\lambda = 2$:

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \ s, t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| \, s, t \in R \right\} : \text{the eigenspace of } A \text{ corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2

- Notes:

  (1) If an eigenvalue λ_1 occurs as a multiple root (k times) for the characteristic polynominal, then λ_1 has multiplicity k

  (2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace. (In Ex. 5, k is 3 and the dimension of its eigenspace is 2)

- Ex 6：Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Sol: Characteristic equation:

$$\left|\lambda I - A\right| = \begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-1)^2(\lambda-2)(\lambda-3) = 0$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

$$(1)\ \lambda_1 = 1 \quad \Rightarrow (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{G.-J.E.}}{\Rightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$

※The dimension of the eigenspace of $\lambda_1 = 1$ is 2

$$(2)\ \lambda_2 = 2 \ \Rightarrow (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{G.-J.E.}}{\Longrightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace corresponding to $\lambda_2 = 2$

※The dimension of the eigenspace of λ_2 = 2 is 1

$$(3)\ \lambda_3 = 3 \ \Rightarrow (\lambda_3 I - A)\mathbf{x} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{G.-J.E.}}{\Longrightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_3 = 3$

※The dimension of the eigenspace of $\lambda_3 = 3$ is 1

- Thm. 3: Eigenvalues for triangular matrices

  If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal

- Ex 7: Finding eigenvalues for triangular and diagonal matrices

$$\text{(a) } A=\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \qquad \text{(b) } A=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$\text{(a) } |\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ -5 & -3 & \lambda+3 \end{vmatrix} = (\lambda-2)(\lambda-1)(\lambda+3)=0$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -3$$

$$\text{(b) } \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 0,\ \lambda_4 = -4,\ \lambda_5 = 3$$

- Ex 8: Finding eigenvalues and eigenvectors for standard matrices

Find the eigenvalues and corresponding eigenvectors for

$$A=\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

※ A is the standard matrix for $T(x_1, x_2, x_3) = (x_1 + 3x_2, 3x_1 + x_2, -2x_3)$ (see Slides 7.19 and 7.20)

Sol:

$$|\lambda I - A| = \begin{bmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{bmatrix} = (\lambda+2)^2(\lambda-4)=0$$

$\Rightarrow$ eigenvalues $\lambda_1 = 4,\ \lambda_2 = -2$

For $\lambda_1 = 4$, the corresponding eigenvector is $(1,\ 1,\ 0)$.

For $\lambda_2 = -2$, the corresponding eigenvectors are $(1,\ -1,\ 0)$ and $(0,\ 0,\ 1)$.

Transformation matrix A' for nonstandard bases

Suppose B is the standard basis of R^n. Since the coordinate matrix of a vector relative to the standard basis consists of the components of that vector, i.e., for any $\mathbf{x}$ in R^n, $\mathbf{x} = [\mathbf{x}]_B$.

$$T(\mathbf{x}) = A\mathbf{x} \Rightarrow [T(\mathbf{x})]_B = A[\mathbf{x}]_B, \text{ where } A = \left[[T(\mathbf{e}_1)]_B \ \ [T(\mathbf{e}_2)]_B \cdots [T(\mathbf{e}_n)]_B \right]$$

is the standard matrix for T or the matrix of T relative to the standard basis B

The above theorem can be extended to consider a nonstandard basis B', which consists of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$$[T(\mathbf{x})]_{B'} = A'[\mathbf{x}]_{B'}, \text{ where } A' = \left[[T(\mathbf{v}_1)]_{B'} \ \ [T(\mathbf{v}_2)]_{B'} \cdots [T(\mathbf{v}_n)]_{B'} \right]$$

is the transformation matrix for T relative to the basis B'

2 Diagonalization

- Diagonalization problem :

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix :

  Definition 1: A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (i.e., P diagonalizes A)

  Definition 2: A square matrix A is called **diagonalizable** if A is **similar** to a diagonal matrix

  ※ In Sec. 6.4, two square matrices A and B are **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$.

- Notes:

  This section shows that the eigenvalue and eigenvector problem is closely related to the diagonalization problem

- Thm. 4: Similar matrices have the same eigenvalues

  If A and B are similar $n\times n$ matrices, then they have the same eigenvalues

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

For any diagonal matrix in the form of $D = \lambda I$, $P^{-1}DP = D$

Consider the characteristic equation of B:

$$\begin{aligned}|\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda IP - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\ &= |\lambda I - A|\end{aligned}$$

Since A and B have the same characteristic equation, they are with the same eigenvalues

※ Note that the eigenvectors of A and B are not necessarily identical

- Ex 1: Eigenvalue problems and diagonalization programs

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{The eigenvalues}: \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\ \lambda=4 \Rightarrow \text{ the eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

(2) $\lambda = -2 \Rightarrow$ the eigenvector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Note: If $P = [\mathbf{p}_2 \quad \mathbf{p}_1 \quad \mathbf{p}_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Thm. 5: Condition for diagonalization

  An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors

※ If there are n linearly independent eigenvectors, it does not imply that there are n distinct eigenvalues. In an extreme case, it is possible to have only one eigenvalue with the multiplicity n, and there are n linearly independent eigenvectors for this eigenvalue

※ On the other hand, if there are n distinct eigenvalues, then there are n linearly independent eigenvectors, and thus A must be diagonalizable

Ex 4: A matrix that is not diagonalizable

Show that the following matrix is not diagonalizable

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

The eigenvalue $\lambda_1=1,$ and then solve $(\lambda_1 I-A)\mathbf{x}=\mathbf{0}$ for eigenvectors

$$\lambda_1 I-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\Rightarrow \text{eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Since A does not have two linearly independent eigenvectors,

A is not diagonalizable

- Steps for diagonalizing an $n\times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Step 2: Let $P = [\mathbf{p}_1 \; \mathbf{p}_2 \; \cdots \; \mathbf{p}_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \quad i = 1, \; 2, \ldots, \; n$

- Ex 5: Diagonalizing a matrix

$$A=\begin{bmatrix}1 & -1 & -1\\ 1 & 3 & 1\\ -3 & 1 & -1\end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & 1 & 1\\ -1 & \lambda-3 & -1\\ 3 & -1 & \lambda+1\end{vmatrix}=(\lambda-2)(\lambda+2)(\lambda-3)=0$$

$$\text{The eigenvalues}: \ \lambda_1=2,\ \lambda_2=-2,\ \lambda_3=3$$

$$\lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2 \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3 \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \text{ and it follows that}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Note: a quick way to calculate A^k based on the diagonalization technique

$$(1)\ D=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} \Rightarrow D^k=\begin{bmatrix}\lambda_1^k & 0 & \cdots & 0\\ 0 & \lambda_2^k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n^k\end{bmatrix}$$

$$(2)\ D=P^{-1}AP \Rightarrow D^k=\underbrace{\underbrace{P^{-1}AP}\,\underbrace{P^{-1}AP}\cdots\underbrace{P^{-1}AP}}_{\text{repeat } k \text{ times}}=P^{-1}A^kP$$

$$A^k=PD^kP^{-1},\ \text{where } D^k=\begin{bmatrix}\lambda_1^k & 0 & \cdots & 0\\ 0 & \lambda_2^k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n^k\end{bmatrix}$$

- Thm. 6: Sufficient conditions for diagonalization

  If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and thus A is diagonalizable.

- Ex 7: Determining whether a matrix is diagonalizable

$$A=\begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & -3\end{bmatrix}$$

Sol: Because A is a triangular matrix, its eigenvalues are

$$\lambda_1=1,\ \lambda_2=0,\ \lambda_3=-3$$

According to Thm. 6, because these three values are distinct, A is diagonalizable

- Ex 8: Finding a diagonalized matrix for a linear transformation

  Let $T: R^3 \rightarrow R^3$ be the linear transformation given by

  $$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

  Find a basis B' for R^3 such that the matrix for T relative to B' is diagonal

Sol:

The standard matrix for T is given by

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From Ex. 5 you know that $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 3$ and thus A is diagonalizable.

$$B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(-1,\ 0,\ 1), (1,\ -1,\ 4), (-1,\ 1,\ 1)\}$$

The matrix for T relative to this basis is

$$A' = \left[[T(\mathbf{v}_1)]_{B'} \quad [T(\mathbf{v}_2)]_{B'} \quad [T(\mathbf{v}_3)]_{B'} \right]$$
$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3 Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix :

A square matrix A is symmetric if it is equal to its transpose:

$$A = A^T$$

- Ex 1: Symmetric matrices and nonsymetric matrices

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$ (symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$ (symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$ (nonsymmetric)

- Thm 7: Eigenvalues of symmetric matrices

  If A is an $n \times n$ "symmetric" matrix, then the following properties are true

  (1) A is diagonalizable (symmetric matrices (except the matrices in the form of $A = aI$, in which case A is already diagonal) are guaranteed to have n linearly independent eigenvectors and thus be diagonalizable)

  (2) All eigenvalues of A are real numbers

  (3) If λ is an eigenvalue of A with the multiplicity to be k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k

※ The above theorem is called the **Real Spectral Theorem**, and the set of eigenvalues of A is called the **spectrum** of A

- Ex 2:

Prove that a 2×2 symmetric matrix is diagonalizable

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Pf: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

As a function in λ, this quadratic polynomial function has a nonnegative discriminant as follows

$$\begin{aligned}(a+b)^2 - 4(1)(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= \underline{(a-b)^2 + 4c^2} \geq 0 \Rightarrow \text{real-number solutions}\end{aligned}$$

(1) $(a-b)^2+4c^2=0$

$\Rightarrow\ a=b,\ c=0$

$A=\begin{bmatrix} a & c \\ c & b \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ itself is a diagonal matrix

※ Note that in this case, A has one eigenvalue, a, whose multiplicity is 2, and the two eigenvectors are linearly independent

(2) $(a-b)^2+4c^2>0$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. According to Thm. 6, A is diagonalizable

- Orthogonal matrix :

  A square matrix P is called orthogonal if it is invertible and

$$P^{-1} = P^T \text{ (or } PP^T = P^T P = I\text{)}$$

- Thm. 8: Properties of orthogonal matrices

  An $n\times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set

Pf: Suppose the column vectors of P form an orthonormal set, i.e.,

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}, \text{ where } \mathbf{p}_i \cdot \mathbf{p}_j = 0 \text{ for } i \neq j \text{ and } \mathbf{p}_i \cdot \mathbf{p}_i = 1$$

$$P^T P = \begin{bmatrix} \mathbf{p}_1{}^T\mathbf{p}_1 & \mathbf{p}_1{}^T\mathbf{p}_2 & \cdots & \mathbf{p}_1{}^T\mathbf{p}_n \\ \mathbf{p}_2{}^T\mathbf{p}_1 & \mathbf{p}_2{}^T\mathbf{p}_2 & \cdots & \mathbf{p}_2{}^T\mathbf{p}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_n{}^T\mathbf{p}_1 & \mathbf{p}_n{}^T\mathbf{p}_2 & \cdots & \mathbf{p}_n{}^T\mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1\cdot\mathbf{p}_1 & \mathbf{p}_1\cdot\mathbf{p}_2 & \cdots & \mathbf{p}_1\cdot\mathbf{p}_n \\ \mathbf{p}_2\cdot\mathbf{p}_1 & \mathbf{p}_2\cdot\mathbf{p}_2 & \cdots & \mathbf{p}_2\cdot\mathbf{p}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_n\cdot\mathbf{p}_1 & \mathbf{p}_n\cdot\mathbf{p}_2 & \cdots & \mathbf{p}_n\cdot\mathbf{p}_n \end{bmatrix} = I_n$$

It implies that $P^{-1} = P^T$ and thus P is orthogonal

- Ex 5: Show that P is an orthogonal matrix.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Sol: If P is a orthogonal matrix, then $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Moreover, let $\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{-2}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3\sqrt{5}} \end{bmatrix}$, and $\mathbf{p}_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$,

we can produce $\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$ and $\mathbf{p}_1 \cdot \mathbf{p}_1 = \mathbf{p}_2 \cdot \mathbf{p}_2 = \mathbf{p}_3 \cdot \mathbf{p}_3 = 1$

So, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set.

- Thm. 9: Properties of symmetric matrices

  Let A be an $n \times n$ "symmetric" matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Pf:

$$\lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) = (\lambda_1\mathbf{x}_1) \cdot \mathbf{x}_2 = (A\mathbf{x}_1) \cdot \mathbf{x}_2 = (A\mathbf{x}_1)^T \mathbf{x}_2 = (\mathbf{x}_1^T A^T)\mathbf{x}_2$$

$$\overset{\text{because } A \text{ is symmetric}}{=} (\mathbf{x}_1^T A)\mathbf{x}_2 = \mathbf{x}_1^T (A\mathbf{x}_2) = \mathbf{x}_1^T (\lambda_2\mathbf{x}_2) = \mathbf{x}_1 \cdot (\lambda_2\mathbf{x}_2) = \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2)$$

The above equation implies $(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0,$ and because $\lambda_1 \neq \lambda_2,$ it follows that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0.$ So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal

※ For distinct eigenvalues of a symmetric matrix, their corresponding eigenvectors are orthogonal and thus linearly independent to each other

※ Note that there may be multiple $\mathbf{x}_1$'s and $\mathbf{x}_2$'s corresponding to λ_1 and λ_2

- Orthogonal diagonalization :

  A matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal

- Thm. 10: Fundamental theorem of symmetric matrices

  An $n \times n$ matrix A is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric

Pf:

($\Rightarrow$)

A is orthogonally diagonalizable

$\Rightarrow D = P^{-1}AP$ is diagonal, and P is an orthogonal matrix s.t. $P^{-1} = P^T$

$$\Rightarrow A = PDP^{-1} = PDP^T \Rightarrow A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

- Orthogonal diagonalization of a symmetric matrix:

Let A be an $n\times n$ symmetric matrix.

(1) Find all eigenvalues of A and determine the multiplicity of each

※ According to Thm. 9, eigenvectors corresponding to distinct eigenvalues are orthognoal

(2) For each eigenvalue of multiplicity 1, choose the unit eigenvector

(3) For each eigenvalue of the multiplicity to be $k \geq 2$, find a set of k linearly independent eigenvectors. If this set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is not orthonormal, apply the Gram-Schmidt orthonormalization process

It is known that G.-S. process is a kind of linear transformation, i.e., the produced vectors can be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ (see Slide 5.55),

i. Since $A\mathbf{v}_1 = \lambda\mathbf{v}_1, A\mathbf{v}_2 = \lambda\mathbf{v}_2, \ldots, A\mathbf{v}_k = \lambda\mathbf{v}_k,$

$\Rightarrow A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = \lambda(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k)$

$\Rightarrow$ The produced vectors through the G.-S. process are still eigenvectors for λ

ii. Since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are orthogonal to eigenvectors corresponding to other different eigenvalues (according to Thm. 7.9), $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is also orthogonal to eigenvectors corresponding to other different eigenvalues.

(4) The composite of steps (2) and (3) produces an orthonormal set of n eigenvectors. Use these orthonormal and thus linearly independent eigenvectors as column vectors to form the matrix P.

i. According to Thm. 8, the matrix P is orthogonal

ii. Following the diagonalization process , $D = P^{-1}AP$ is diagonal

Therefore, the matrix A is orthogonally diagonalizable

- Ex 7: Determining whether a matrix is orthogonally diagonalizable

	Symmetric matrix	Orthogonally diagonalizable
$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	○	○
$A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$	×	×
$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	×	×
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$	○	○

- Ex 9: Orthogonal diagonalization

Find an orthogonal matrix P that diagonalizes A.

$$A=\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

Sol:

(1) $|\lambda I - A| = (\lambda-3)^2(\lambda+6)=0$

$\lambda_1=-6,\ \lambda_2=3$ (has a multiplicity of 2)

(2) $\lambda_1=-6,\ \mathbf{v}_1=(1,\ -2,\ 2) \;\Rightarrow\; \mathbf{u}_1=\dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}=(\frac{1}{3},\ \frac{-2}{3},\ \frac{2}{3})$

(3) $\lambda_2=3,\ \mathbf{v}_2=(2,\ 1,\ 0),\ \mathbf{v}_3=(-2,\ 4,\ 5)$

orthogonal

INFINITE SERIES

Introduction; Definition

To form an infinite series, we begin with an infinite sequence of real numbers: $a_0, a_1, a_2, \ldots$. We can't form the sum of all the a_k (there are an infinite number of them), but we can form the *partial sums*:

$$s_0 = a_0 = \sum_{k=0}^{0} a_k,$$

$$s_1 = a_0 + a_1 = \sum_{k=0}^{1} a_k,$$

$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^{2} a_k,$$

$$s_3 = a_0 + a_1 + a_2 + a_3 = \sum_{k=0}^{3} a_k,$$

$$s_n = a_0 + a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

and so on.

If, as $n \to \infty$, the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

tends to a finite limit L, we write

$$\sum_{k=0}^{\infty} a_k = L$$

and say that

$$\text{the series} \quad \sum_{k=0}^{\infty} a_k \quad \textit{converges to } L.$$

We call L the *sum* of the series. If the sequence of partial sums diverges, we say that

$$\text{the series} \quad \sum_{k=0}^{\infty} a_k \quad \textit{diverges}.$$

Example

The series

$$\sum_{k=0}^{\infty}(-1)^k \quad \text{and} \quad \sum_{k=0}^{\infty}2^k$$

illustrate two forms of divergence: *bounded divergence, unbounded divergence.*

For the first series,

$$s_n = 1 - 1 + 1 - 1 + \cdots + (-1)^n.$$

Here

$$s_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

The sequence of partial sums reduces to 1, 0, 1, 0, … . Since the sequence diverges, the series diverges. This is an example of bounded divergence.

For the second series,

$$s_n = \sum_{k=0}^{n}2^k = 1 + 2 + 2^2 + \cdots + 2^n$$

Since $s_n > 2^n$, the sum tends to ∞, and the series diverges. This is an example of unbounded divergence

The Geometric Series

The *geometric progression*

$$1, x, x^2, x^3, \dots$$

gives rise to the numbers

$$1, 1+x, 1+x+x^2, 1+x+x^2+x^3, \dots .$$

These numbers are the partial sums of what is called the *geometric series*:

$$\sum_{k=0}^{\infty} x^k$$

This series is so important that we will give it special attention.

(i) If $|x| < 1$, then $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

(ii) If $|x| \geq 1$, then $\sum_{k=0}^{\infty} x^k$ diverges.

1. If $\sum_{k=0}^{\infty} a_k$ converges and $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} (a_k + b_k)$ converges.

Moreover, if $\sum_{k=0}^{\infty} a_k = L$ and $\sum_{k=0}^{\infty} b_k = M$, then $\sum_{k=0}^{\infty} (a_k + b_k) = L + M$.

2. If $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} \alpha\, a_k$ converges for each real number α.

Moreover, if $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha\, a_k = \alpha\, L$.

The *kth term* of a convergent series tends to 0; namely,

$$\text{if} \quad \sum_{k=0}^{\infty} a_k \quad \text{converges,} \quad \text{then} \quad a_k \to 0 \quad \text{as} \quad k \to \infty.$$

$$\text{if} \quad a_k \not\to 0 \quad \text{then} \quad \sum_{k=0}^{\infty} a_k \quad \text{diverges.}$$

THE INTEGRAL TEST

If f is continuous, positive, and decreasing on $[1, \infty)$, then

$$\sum_{k=1}^{\infty} f(k) \quad \text{converges} \quad \text{iff} \quad \int_1^{\infty} f(x)\,dx \quad \text{converges.}$$

(*The harmonic series*)

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{diverges.}}$$

(*The p-series*)

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad \text{converges} \qquad \text{iff} \quad p > 1.}$$

THE BASIC COMPARISON THEOREM

Suppose that $\sum a_k$ and $\sum b_k$ are series with nonnegative terms and

$$\sum a_k \le \sum b_k \qquad \text{for all } k \text{ sufficiently large.}$$

(i) If $\sum b_k$ converges, then $\sum a_k$ converges.

(ii) If $\sum a_k$ diverges, then $\sum b_k$ diverges.

Applying the Basic Comparison Theorem

Example

(a) $\sum \frac{1}{2k^3+1}$ converges by comparison with $\sum \frac{1}{k^3}$

$$\frac{1}{2k^3+1} < \frac{1}{k^3} \quad \text{and} \quad \sum \frac{1}{k^3} \quad \text{converges}$$

(b) $\sum \frac{k^3}{k^5+5k^4+7}$ converges by comparison with $\sum \frac{1}{k^2}$

$$\frac{k^3}{k^5+5k^4+7} < \frac{k^3}{k^5} = \frac{1}{k^2} \quad \text{and} \quad \sum \frac{1}{k^2} \quad \text{converges}$$

THE LIMIT COMPARISON THEOREM

Let $\sum a_k$ and $\sum b_k$ be series with *positive terms*. If $a_k/b_k \to L$, and L is *positive*, then

$$\sum a_k \quad \text{converges} \quad \text{iff} \quad \sum b_k \quad \text{converges}$$

Example

Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{5^k - 3}$ converges or diverges.

Solution

For large k

$$\frac{1}{5^k - 3} \quad \text{differs little from} \quad \frac{1}{5^k}$$

As $k \to \infty$,

$$\frac{1}{5^k - 3} \div \frac{1}{5^k} = \frac{5^k}{5^k - 3} = \frac{1}{1 - 3/5^k} \to 1$$

Since

$$\sum \frac{1}{5^k} \quad \text{converges,}$$

(it is a convergent geometric series), the original series converges.

THE ROOT TEST

Let $\sum a_k$ be a series with nonnegative terms, and suppose that

$$(a_k)^{1/k} \to \rho.$$

(i) If $\rho < 1$, then $\sum a_k$ converges.

(ii) If $\rho > 1$, then $\sum a_k$ diverges.

(iii) If $\rho = 1$, then the test is inconclusive. The series may converge; it may diverge.

Applying the Root Test

Example

For the series $\sum \frac{1}{(\ln k)^k}$

$$(a_k)^{1/k} = \frac{1}{\ln k} \to 0$$

The series converges.

THE RATIO TEST

Let $\sum a_k$ be a series with positive terms and suppose that

$$\frac{a_{k+1}}{a_k} \to \lambda.$$

(i) If $\lambda < 1$, then $\sum a_k$ converges.

(ii) If $\lambda > 1$, then $\sum a_k$ diverges.

(iii) If $\lambda = 1$, then the test is inconclusive. The series may converge, it may diverge.

Applying the Ratio Test

Example

For the series $\sum \frac{k}{10^k}$

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{10^{k+1}} \cdot \frac{10^k}{k} = \frac{1}{10}\frac{k+1}{k} \to \frac{1}{10}$$

The series converges.

Summary on Convergence Tests

In general, the root test is used only if powers are involved. The ratio test is particularly effective with factorials and with combinations of powers and factorials. If the terms are rational functions of k, the ratio test is inconclusive and the root test is difficult to apply. Series with rational terms are most easily handled by limit comparison with a *p*-series, a series of the form $\sum 1/k^p$. If the terms have the configuration of a derivative, you may be able to apply the integral test. Finally, keep in mind that, if $a_k \not\to 0$, then there is no reason to try any convergence test; the series diverges.

Absolute and Conditional Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Series Σa_k for which $\Sigma|a_k|$ converges are called *absolutely convergent*. The theorem we have just proved says that

$$\boxed{\text{absolutely convergent series are convergent.}}$$

Absolute and Conditional Convergence

Alternating Series

A series such as

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

in which consecutive terms have opposite signs is a called an *alternating series*. As in our example, we shall follow custom and begin all alternating series with a positive term. In general, then, an alternating series will look like this.

$$a_0-a_1+a_2-a_3+\cdots=\sum_{k=0}^{\infty}(-1)^k a_k$$

with all the a_k positive. In this setup the partial sums of even index end with a positive term and the partial sums of odd index end with a negative term.

THE BASIC THEOREM ON ALTERNATING SERIES

Let $a_0, a_1, a_2,$ be a decreasing sequence of positive numbers. The series

$$a_0 - a_1 + a_2 - a_3 + \cdots = \sum_{k=0}^{\infty} (-1)^k a_k \quad \text{converges} \quad \text{iff} \quad a_k \to 0$$

Estimating the Sum of an Alternating Series

You have seen that if $a_0, a_1, a_2, \ldots$ is a decreasing sequence of positive numbers that tends to 0, then

$$\sum_{k=0}^{\infty} (-1)^k a_k \quad \text{converges to some sum } L.$$

The number L lies between all consecutive partial sums, s_n, s_{n+1}. From this it follows that s_n approximates L to within a_{n+1}:

$$|s_n - L| < a_{n+1}.$$

TAYLOR'S THEOREM

If f has $n+1$ continuous derivatives on an open interval I that contains 0, then for each $x \in I$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

with $R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$. We call $R_n(x)$ the *remainder*.

LAGRANGE FORMULA FOR THE REMAINDER

The remainder in Taylor's theorem can be written

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

with c some number between 0 and x.

Taylor Polynomials in x; Taylor Series in x

Taylor Series in x

By definition $0! = 1$. Adopting the convention that $f^{(0)} = f$, we can write Taylor polynomials

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

in Σ notation:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

In this case, we say that $f(x)$ can be expanded as a *Taylor series in x* and write

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all real } x.$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all real } x.$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all real } x.$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \text{for } -1 < x \le 1.$$

TAYLOR'S THEOREM

If g has $n + 1$ continuous derivatives on an open interval I that contains the point a, then for each $x \in I$

$$g(x) = g(a) + g'(a)(x - a) + \frac{g''(a)}{2!}(x - a)^2 + \cdots + \frac{g^{(n)}(a)}{n!}(x - a)^n + R_n(x)$$

with

$$R_n(x) = \frac{1}{n!}\int_a^x g^{(n+1)}(t)(x - t)^n\,dt.$$

Taylor Polynomials and Taylor Series in $x - a$

The polynomial

$$P_n(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n$$

is called the *nth Taylor polynomial for* g *in powers of* $x - a$. In this more general setting, the *Lagrange formula for the remainder*, $R_n(x)$, takes the form

$$\boxed{R_n(x) = \frac{g^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

where c is some number between a and x.
Now let $x \in I$, $x \neq a$, and let J be the closed interval that joins a to x. Then

$$\boxed{|R_n(x)| \leq \left(\max_{t \in J} |g^{(n+1)}(t)|\right) \frac{|x-a|^{n+1}}{(n+1)!}}$$

If $R_n(x) \to 0$, then we have the series representation

$$P_n(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + \cdots,$$

which in sigma notation, takes the form

$$\boxed{g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(x-a)^k}$$

This is known as the Taylor expansion of $g(x)$ in powers of $x - a$. The series on the right is called a *Taylor series in* $x - a$.

Power Series

A power series $\sum a_k x^k$ is said to converge

(i) at c if $\sum a_k c^k$ converges;

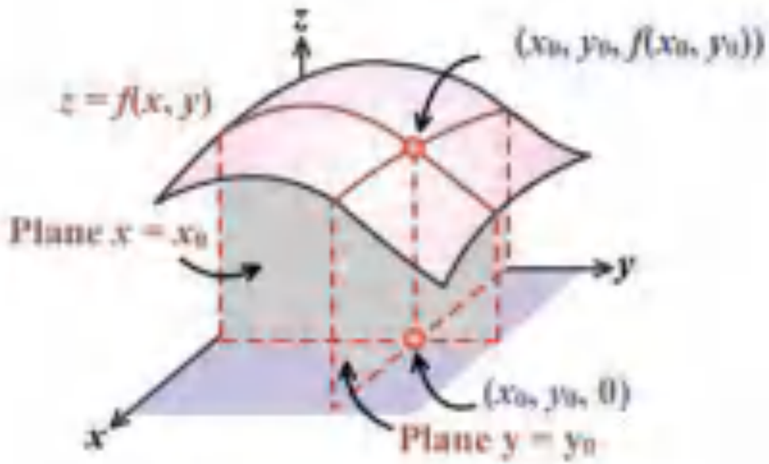
(ii) on the set S if $\sum a_k x^k$ converges at each $x \in S$.

PARTIAL DIFFERENTIATION

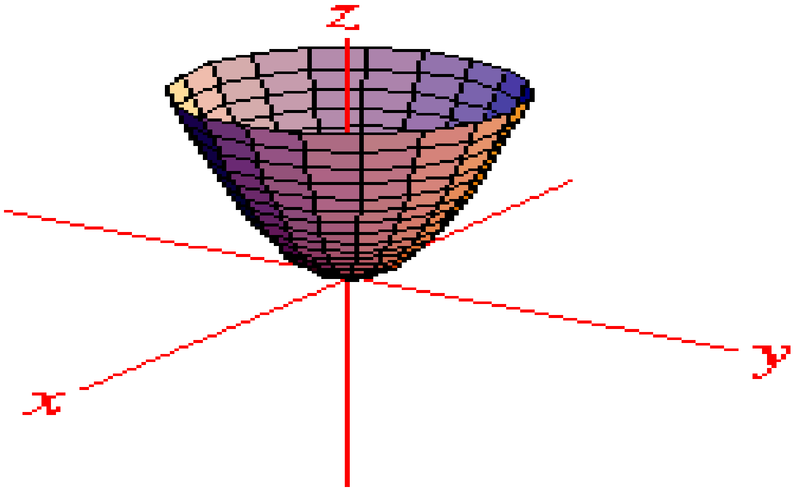
IMPORTANT TERMS, DEFINITIONS & RESULTS

01. *Function of Two Variables*

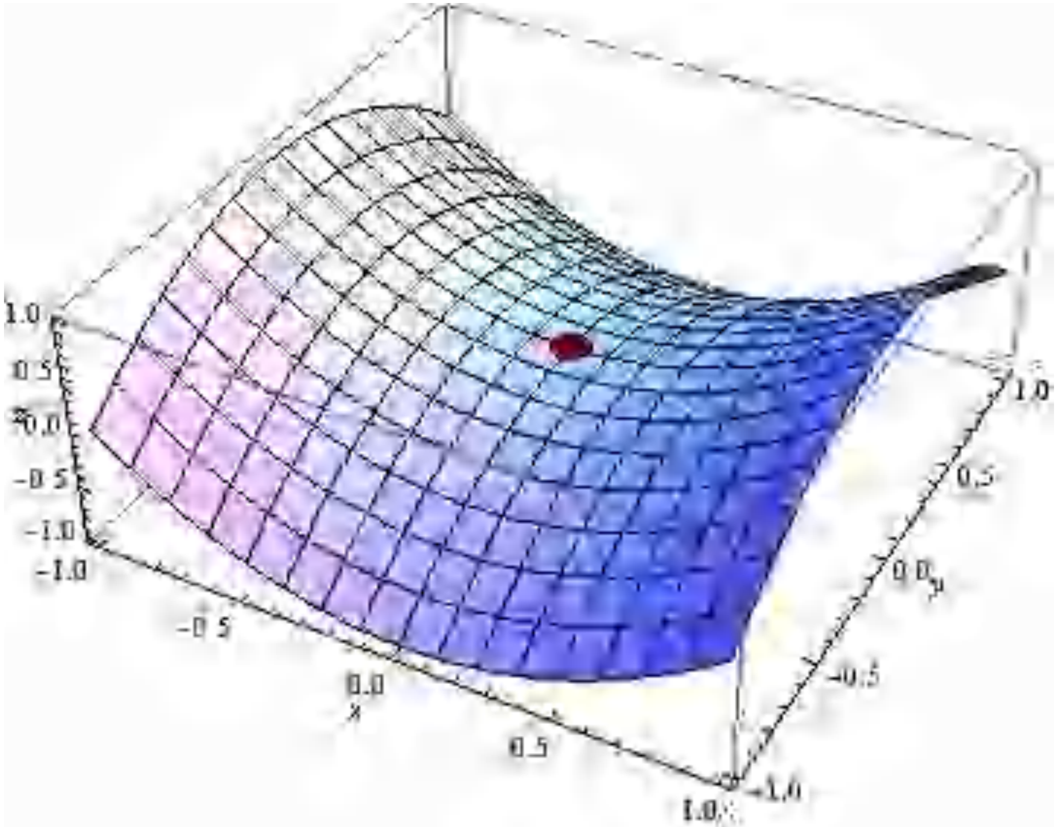
Let us consider a set of points D in the plane. A rule which assigns a unique real value z to each point in D is called a real valued function defined on D. Using rectangular coordinates, we can identify every point on the plane by an ordered pair (x, y) of real numbers. Thus the real valued function may be represented as $z = f(x, y)$ where x and y are called independent variables and z is the dependent variable. D is known as the domain of the function f.


z
$(x_0, y_0, f(x_0, y_0))$
$z = f(x, y)$
Plane $x = x_0$
y
$(x_0, y_0, 0)$
x
Plane $y = y_0$

Geometrically, given a function f of two independent variables x and y, we may consider the graph of $z=f(x,y)$ in the xyz- space. It defines a surface S spanning over the domain D in the xy-plane. For instance, the graph of the function $z=x^2+y^2$ is a paraboloid, while the graph of $z=x^2-y^2$ has the shape of a saddle.



$$z = x^2 + y^2$$


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
x
0.5
1.0
−1.0
−0.5
0.0
y
0.5
1.0

03. *Continuity*:

A real valued function $z = f(x, y)$ defined on a region D of the xy −plane is said to be continuous at a point $(a,b) \in D$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ *i.e.* the limit of f as (x,y) approaches (a,b) is equal to the value of f at (a,b).

If f is continuous at every point in D, it is said to be continuous on D.

Note: Let $f(x,y)$ and $g(x,y)$ be continuous functions at $(a,b)\in D$. Then

1. $f+g$, $f-g$ and fg are continuous functions at (a,b).

2. $\dfrac{f}{g}$ is continuous at (a,b) if $g(a,b)\neq 0$.

3. If h is a single-variable function continuous at $f(a,b)$ then the composite function hof defined by $hof(x,y)=h(f(x,y))$ is continuous at (a,b)

04 Partial Differentiation:

A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are held constant. **Partial differentiation** is the process of finding partial derivatives. All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) with respect to one variable, all the other variables are treated (temporarily) as constants.

Let $z = f(x, y)$ be a function of two variables. The x – partial derivative (or x – derivative) and y – partial derivative (or y – derivative) of $z = f(x, y)$ at (x, y) are the limits

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided these limits exist and are finite.

Notation: The partial derivative $\frac{\partial z}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or $f_x(x,y)$ or $D_x f$ or $f_1(x,y)$ where the subscripts x and 1 denote the variable w.r.t. which the partial differentiation is carried out.

Remark: Partial derivatives for functions of three or more variables can be dealt with similarly. For instance, the partial derivative of a function of three variables $u = f(x,y,z)$ at a point (x_0, y_0, z_0) with respect to x is defined by

$$f_x(x_0, y_0, z_0) = \left(\frac{\partial u}{\partial x}\right)_{(x_0, y_0, z_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

The other two partial derivatives are

$$f_y(x_0, y_0, z_0) = \left(\frac{\partial u}{\partial y}\right)_{(x_0, y_0, z_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y, z_0) - f(x_0, y_0, z_0)}{\Delta y}$$

$$f_z(x_0, y_0, z_0) = \left(\frac{\partial u}{\partial x}\right)_{(x_0, y_0, z_0)} = \lim_{\Delta z \to 0} \frac{f(x_0, y_0, z_0 + \Delta z) - f(x_0, y_0, z_0)}{\Delta z}$$

Homogeneous functions:

- A function $f(x,y)$ of two variables x & y is said to be homogeneous function of degree n, if it can be expressed as

$$f(x,y)=x^n\phi_1\left(\frac{y}{x}\right)\ or\ y^n\phi_2\left(\frac{x}{y}\right)$$

Or

A function $f(x,y)$ is said to be homogeneous function of degree n. If

$f(tx,ty)=t^n f(x,y)$

- A function $f(x,y,z)$ of three variables x,y & z is said to be homogeneous function of degree n, if it can be expressed as

$$f(x,y,z)=x^n\phi_1\left(\frac{y}{x},\frac{z}{x}\right)\ or\ y^n\phi_2\left(\frac{x}{y},\frac{z}{y}\right)\ or\ z^n\phi_3\left(\frac{x}{z},\frac{y}{z}\right)$$

e.g. (1) $f(x,y)=\dfrac{x^2+y^2}{x-y}$ is homogeneous of degree 1.

(2) $f(x,y)=\dfrac{x^2+y}{x-y}$ is not homogeneous function

Euler's Theorem for homogeneous functions

Statement:

If u is homogeneous function of degree n in two variables x & y, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Proof:

Let $\quad u = f(x, y)$

$$= x^n \phi\left(\frac{y}{x}\right) \quad ....(1) \qquad \{\text{As is homogeneous function of degree n}\}$$

Differentiating partially equation (1) w. r. t. x, we get

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\frac{-y}{x^2}$$

$$= nx^{n-1}\phi\left(\frac{y}{x}\right) - yx^{n-2}\phi'\left(\frac{y}{x}\right)$$

Multiply by x, we get

$$x\frac{\partial u}{\partial x} = nx^n \phi\left(\frac{y}{x}\right) - yx^{n-1}\phi'\left(\frac{y}{x}\right) \qquad ....(2)$$

Again differentiating partially equation (1) w. r. t. y, we get

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1}\phi'\left(\frac{y}{x}\right)$$

Multiply by y, we get

$$y\frac{\partial u}{\partial y} = yx^{n-1}\phi'\left(\frac{y}{x}\right) \qquad \text{..................(3)}$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) = nu$$

Hence proved.

If u is homogeneous function of degree n in two variables x & y, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

Proof:

As u is homogeneous function of degree n in two variables x & y, therefore by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{..................(1)}$$

Differentiating partially equation (1) w. r. t. x, **we get**

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

$$\text{or} \qquad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$$

Multiply by x, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (n-1)x\frac{\partial u}{\partial x} \qquad \text{..................(2)}$$

Again Differentiating partially equation (1) w. r. t. y, we get

$$x\frac{\partial^2 u}{\partial y \partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}$$

or
$$x\frac{\partial^2 u}{\partial y \partial x}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$$

Multiply by y, we get

$$xy\frac{\partial^2 u}{\partial y \partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)y\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots(3)$$

Adding equations (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \qquad \left\{\because \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right\}$$

$$=n(n-1)u \qquad \{\text{ using equation (1)}\}$$

Hence proved.

Composite Functions

If $u = f(x, y)$, where $x = \phi_1(t)$, $y = \phi_2(t)$, then u is called a composite function of the **single variable** t and we can obtain $\dfrac{du}{dt}$ which is called the total derivative of u.

If $u = f(x, y)$, where $x = \phi_1(r, s)$, $y = \phi_2(r, s)$, then u is called a composite function of **two variables** r and s and we can obtain $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$.

Differentiation of Composite Functions:

A. If u is a composite function of t, defined by the relation $u = f(x, y)$ and $x = \phi_1(t)$, $y = \phi_2(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

B. If u is a composite function of t, defined by the relation $u = f(x, y, z)$ and $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

C. If u is a composite function of r and s, defined by the relation $u = f(x, y)$ and $x = \phi_1(r, s)$, $y = \phi_2(r, s)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

D. If $u = f(x, y)$, where $y = \phi(x)$, then as $u = \psi(x)$, therefore u is a composite functions of x, so we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

E. If $u = f(x, y)$, where $x = \phi(y)$, then as $u = \psi(y)$, therefore u is a composite functions of y, so we have

$$\frac{du}{dy} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial u}{\partial y}$$

Differentiation of Implicit Function:

When implicit function $f(x,y)=c$ is given then

$$\frac{dy}{dx}=-\frac{f_x}{f_y}$$

$$\frac{d^2y}{dx^2}=-\left[\frac{f_{xx}f_y^2-2f_xf_yf_{xy}+f_{yy}f_x^2}{(f_y)^3}\right]$$

MULTIPLE INTEGRALS

IMPORTANT TERMS, DEFINITIONS & RESULTS

07 *Double Integrals:*

The notion of a double integral is an extension of the concept of a definite integral on the real line to the case of two dimensional space. Let $f(x,y)$ be a continuous function of two independent variables x and y inside and on the boundary of a region R. Divide the region R into subdomains $R_1, R_2, R_3, \ldots, R_n$ of areas $\delta R_1, \delta R_2, \delta R_3, \ldots, \delta R_n$ respectively. Let (x_i, y_i) be an arbitrary point inside the ith elementary area δR_i. Consider the sum

$$S_n = f(x_1,y_1)\delta R_1 + f(x_2,y_2)\delta R_2 + \ldots + f(x_i,y_i)\delta R_i + \ldots + f(x_n,y_n)\delta R_n = \sum_{i=1}^{n} f(x_i,y_i)\delta R_i$$

When $n \to \infty$, the number of sub-regions increases indefinitely such that the largest of the areas δR approaches zero. The $\lim\limits_{\delta R \to 0} S_n$ if exists, is called the ***double integral*** of the function $f(x,y)$ over the region (domain) R and is denoted by $\iint\limits_R f(x,y)\,dR$

If the region R is divided into rectangular meshes by a network of lines parallel to the coordinate axes and if dx and dy be the length and breadth of a rectangular mesh, then $dxdy$ is an element of area in Cartesian coordinates. In such a case, we have $\iint\limits_R f(x,y)\,dR = \iint\limits_R f(x,y)\,dxdy$

or $\iint\limits_R f(x,y)\,dR = \iint\limits_R f(x,y)\,dydx$

08. *Properties of a Double Integral*.

(a) Let $k \neq 0$ be any real number, then $\iint_R kf(x,y)\,dxdy = k\iint_R f(x,y)\,dxdy$

(b) The double integral of the algebraic sum of a finite number of functions f_i is equal to the sum of the double integrals taken for each function. Thus,

$$\iint_R \left[f_1(x,y) + f_2(x,y) + \dots + f_n(x,y) \right] dxdy = \iint_R f_1(x,y)\,dxdy + \iint_R f_2(x,y)\,dxdy + \dots + \iint_R f_n(x,y)\,dxdy$$

(c) $\iint_R f(x,y)\,dxdy = \iint_{R_1} f(x,y)\,dxdy + \iint_{R_2} f(x,y)\,dxdy$, where R is the union of disjoint domains R_1 & R_2 .

09. Evaluation of a double integral:

A double integral can be evaluated by successive single integrations i.e. as a two-fold iterated (repeated) integral as follows (if R is regular in y-direction):

$$I_R = \int_{x=a}^{x=b} \left\{ \int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy \right\} dx \qquad (1)$$

Where the integration is performed first with respect to y (within the braces). With the substitution of the limits $y_1(x)$ and $y_2(x)$, the integrand becomes a function of x alone, which is then integrated with respect to x from a to b.

In a similar way, for a domain R (regular in x-direction) which is bounded above by $x = x_2(y)$ and bounded below by $x = x_1(y)$ and the Abscissa $y = d$ and $y = c$. The double integral is evaluated as

$$I_R = \int_{y=c}^{y=d} \left\{ \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx \right\} dy \qquad (2)$$

In this case the integration is first performed with respect to x and then later with respect to y.

Note: ***(a)*** The braces in (1) and (2) can be omitted since it is conventional to integrate first with respect to a variable whose differential appears first [for example in (1) the order of integration is first y and later x].

(b) When R is regular (in both x and y-directions), draw a rough sketch of R, the domain of integration, to fix the limits of integration. Choose (or change) the order of integration and use (1) or (2) whichever is easier for integration.

10. *Change of order of integration*

As already discussed, for the double integral with variable limits

$$I_R = \iint_R f(x, y)\,dA \qquad (1)$$

The limits of integration can be fixed from a rough sketch of the domain of integration. Then (1) can be evaluated as a two-fold iterated integral using either

$$I_R = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x, y)\,dy\,dx \qquad (2)$$

or

$$I_R = \int_{c}^{d} \int_{g_1(y)}^{g_2(y)} f(x, y)\,dx\,dy \qquad (3)$$

In each specific problem, depending upon the type of the domain R and/or the nature of the integrand, choose either (2) or (3) whichever is easier to evaluate. Thus, in several problems, the evaluation of double integral becomes easier with the change of order of integration, which of course, changes the limits of integration also.

11. *General change of variables in Double Integrals:*

In several cases the evaluation of double integrals becomes easy when we change variables

Let R be region in xy-plane and let x, y be the rectangular Cartesian coordinates of any point P in R. Let u, v be new variables in region R^* such that x, y and u, v are connected through the continuous functions (transformations)

$$x = g(u,v),\quad y = h(u,v)$$

Then u, v are said to be curvilinear coordinates of point P^* in R^* which uniquely corresponds to P in R. Then a given double integral in the old variables x and y can be transformed to a double integral in terms of new variables u, v as follows:

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} F(u,v)\,|J|\,dudv$$

Where J is Jacobian defined as $J = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

❖ *Double Integrals in Polar coordinates:*

For a double integral in Cartesian coordinates x, y, the change of variables to polar coordinates r, θ can be done through the transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

The Jacobian in this case is $J = J\left(\frac{x,y}{r,\theta}\right) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

Therefore, the double integral in Cartesian coordinates x, y gets transformed to double integral in polar coordinated as follows:

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,drd\theta$$

Where R^* is the corresponding domain in polar coordinates

12. Area of plane region:

The area A of a plane region (domain) R is given by

$$A = \iint_R ds = \iint_R dx\,dy = \iint_R dy\,dx$$

Area in polar coordinates is given by $A = \iint_R r\,dr\,d\theta$

1.7. *Triple Integrals:*

Triple integral is a generalization of a double integral. Let V be a given three dimensional domain in space, bounded by closed surface S. Let $f(x, y, z)$ be a continuous function in V of rectangular coordinates x, y, z.

Divide V into subdomains Δv_i. Let $f(P_i)$ be the value of f at an arbitrary point P_i of Δv_i. Then a triple integral of f over the domain V, denoted by $\iiint_V f(P)\,dV$, is defined as

$$\lim \sum f(P_i)\Delta v_i = \iiint_V f(P)\,dV = \iiint_V f(P)\,dx\,dy\,dz$$

14. *General change of variables in Triple Integrals*

Let the functions $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$ be the transformations from cartesian coordinates x, y, z to the curvilinear coordinates u, v, w. Let $F(x,y,z)$ be a continuous function defined in a domain V in the xyz coordinate system. Then a triple integral in cartesian coordinates x, y, z can be transferred to a triple integral in the curvilinear coordinates u, v, w as follows:

$$\iiint_V F(x,y,z)\,dx\,dy\,dz = \iiint_{V'} G(u,v,w)\,|J|\,du\,dv\,dw$$

Where J is Jacobian defined as $J\left(\frac{x,y,z}{u,v,w}\right) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ and V' is the corresponding domain in the curvilinear coordinates u, v, w

❖ ***Triple Integral in Cylindrical Coordinates:***

Cylindrical coordinates r, θ, z are particularly useful in problems of solids having axis of symmetry. The transformation of Cartesian coordinates x, y, z in term of cylindrical coordinates is given by

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$ and the Jacobian in this case is given by

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Therefore, $\iiint\limits_{V} F(x,y,z)\,dx\,dy\,dz = \iiint\limits_{V^*} F(r\cos\theta, r\sin\theta, z)\,|J|\,dr\,d\theta\,dz$

$$= \iiint\limits_{V^*} F(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

❖ *Triple Integral in Spherical Polar Coordinates:*

In problems having symmetry with respect to a point O (generally the origin), it would be convenient to use spherical coordinates with this point chosen as origin. Coordinate transformation from x, y, z to the spherical coordinates r, θ, ϕ are given by

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

and the Jacobian in this case is given by

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & -r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Thus, $\iiint\limits_{V} F(x,y,z)\,dx\,dy\,dz = \iiint\limits_{V'} F(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$

$$= \iiint\limits_{V'} F(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

15. *Volume of a solid:*

Volume of a solid contained in the domain V is given by the triple integral as follows:

$$V = \iiint_V dV = \iiint_V dx\, dy\, dz$$

IMPROPER INTEGRALS

In defining a definite integral $\int_a^b f(x)\,dx$, we dealt with a function *f* defined on a finite interval [*a*, *b*] and we assumed that *f* does not have an infinite discontinuity

Improper Integrals

In this section, we will learn:

How to solve definite integrals

where the interval is infinite and

where the function has an infinite discontinuity.

In this section, we extend the concept of a definite integral to the cases where:

- The interval is infinite
- *f* has an infinite discontinuity in [*a*, *b*]

In either case, the integral is called an improper integral.

- One of the most important applications of this idea, probability distributions, will be studied in Section 8.5

Consider the infinite region *S* that lies:

- Under the curve $y = 1/x^2$
- Above the x-axis
- To the right of the line $x = 1$

INFINITE INTERVALS

You might think that, since *S* is infinite in extent, its area must be infinite.

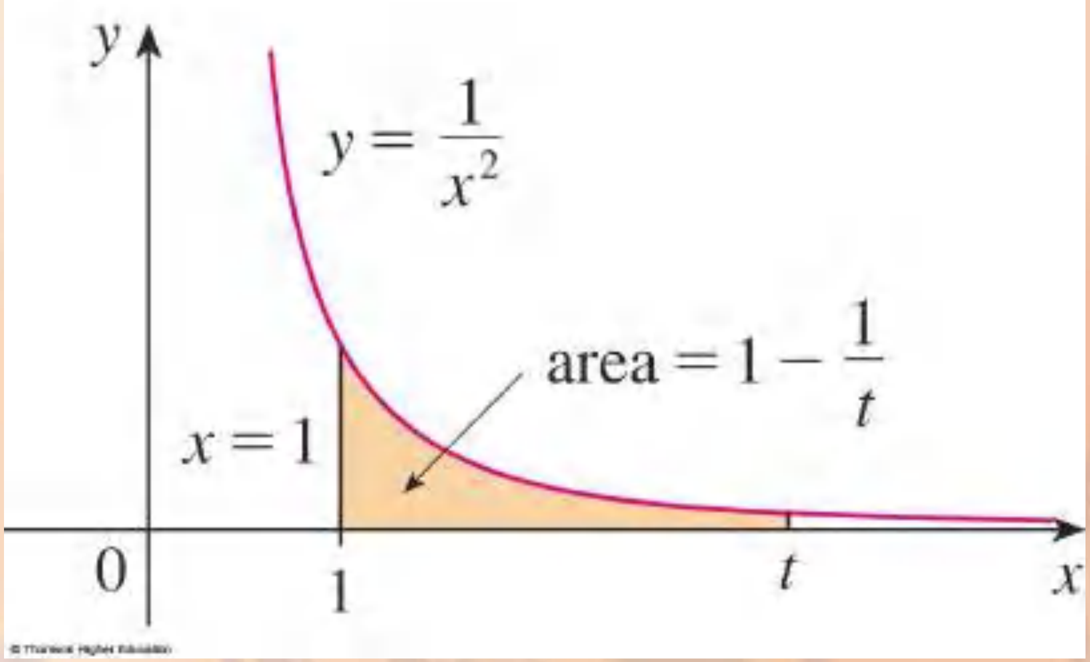
- However, let's take a closer look.

INFINITE INTERVALS

The area of the part of *S* that lies to the left of the line *x* = *t* (shaded) is:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

- Notice that *A*(*t*) < 1 no matter how large *t* is chosen.

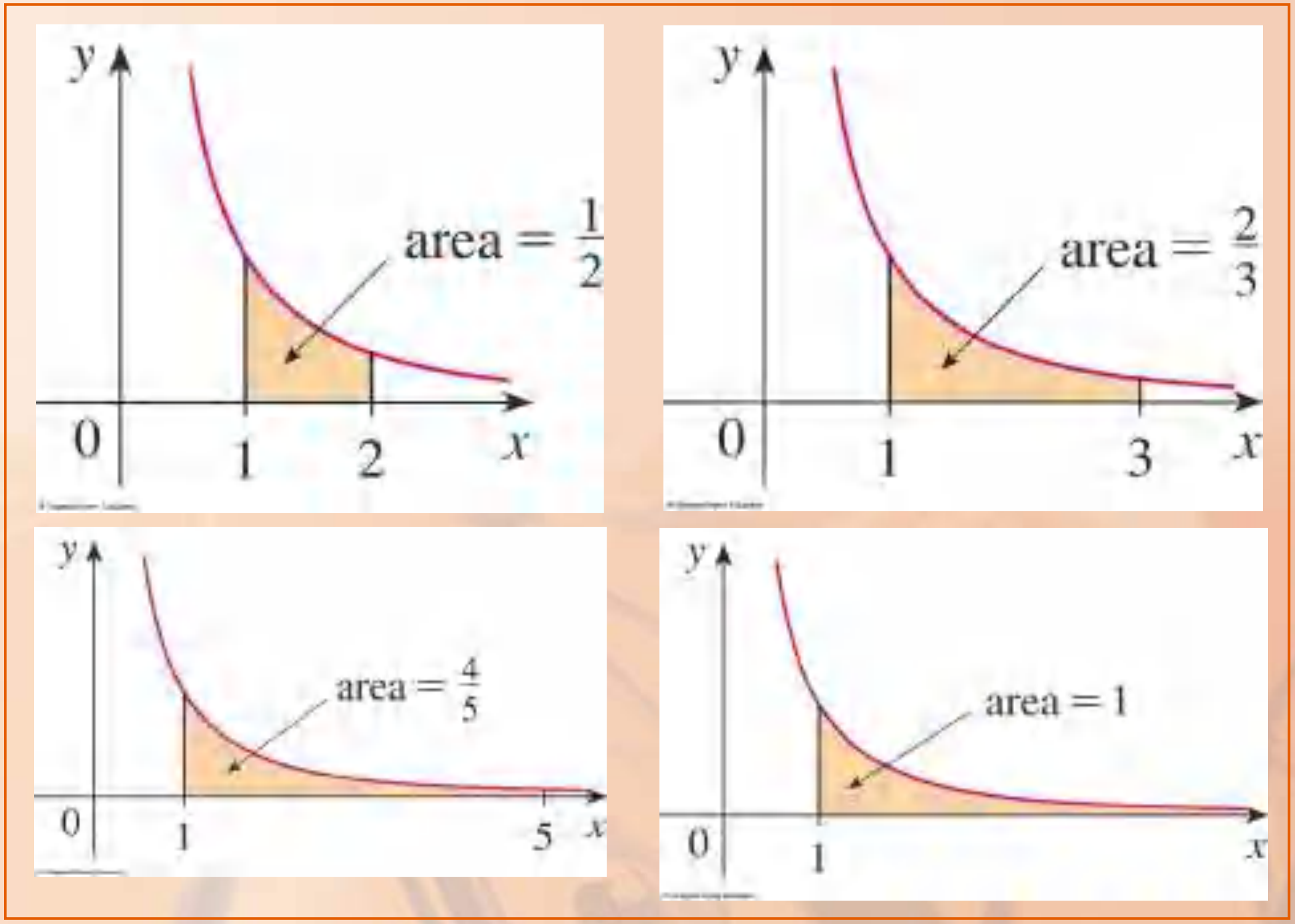


We also observe that:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$$

INFINITE INTERVALS

The area of the shaded region approaches 1 as $t \to \infty$.



So, we say that the area of the infinite region *S* is equal to 1 and we write:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx = 1$$

INFINITE INTERVALS

Using this example as a guide, we define the integral of *f* (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 a**

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 b**

If $\int_t^b f(x)\,dx$ exists for every number $t \leq a$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to \infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

CONVERGENT AND DIVERGENT

Definition 1 b

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 c**

If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

- Here, any real number *a* can be used.

IMPROPER INTEGRALS OF TYPE 1

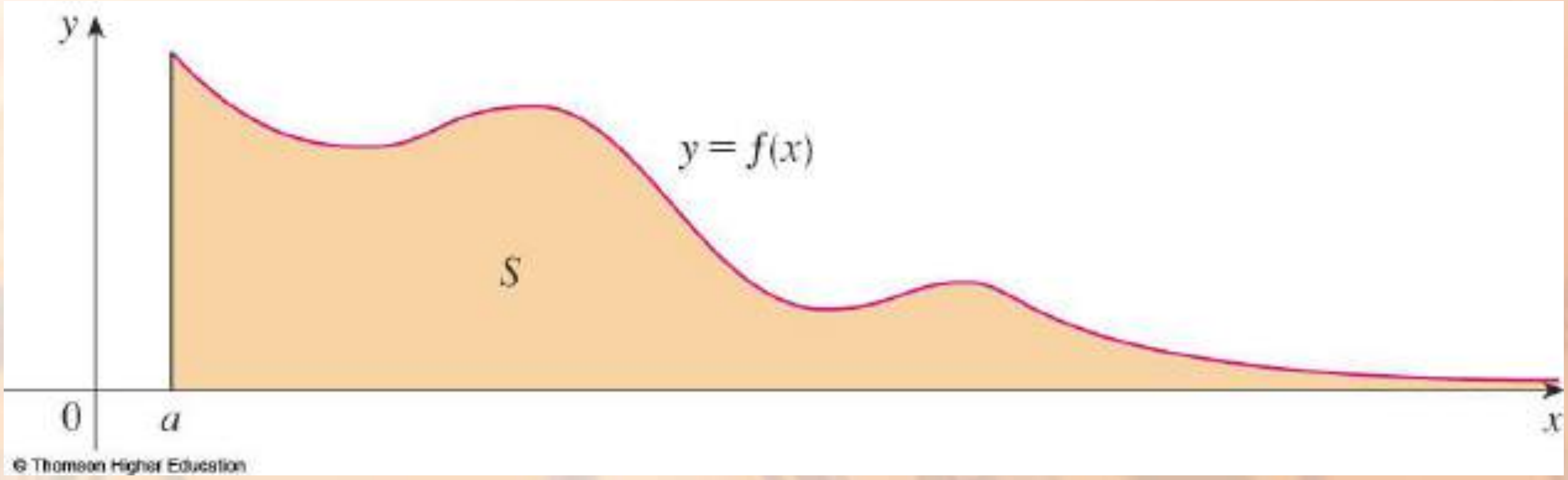
Any of the improper integrals in Definition 1 can be interpreted as an area provided *f* is a positive function.

IMPROPER INTEGRALS OF TYPE 1

For instance, in case (a), suppose $f(x) \geq 0$ and the integral $\int_a^{\infty} f(x)\,dx$ is convergent.

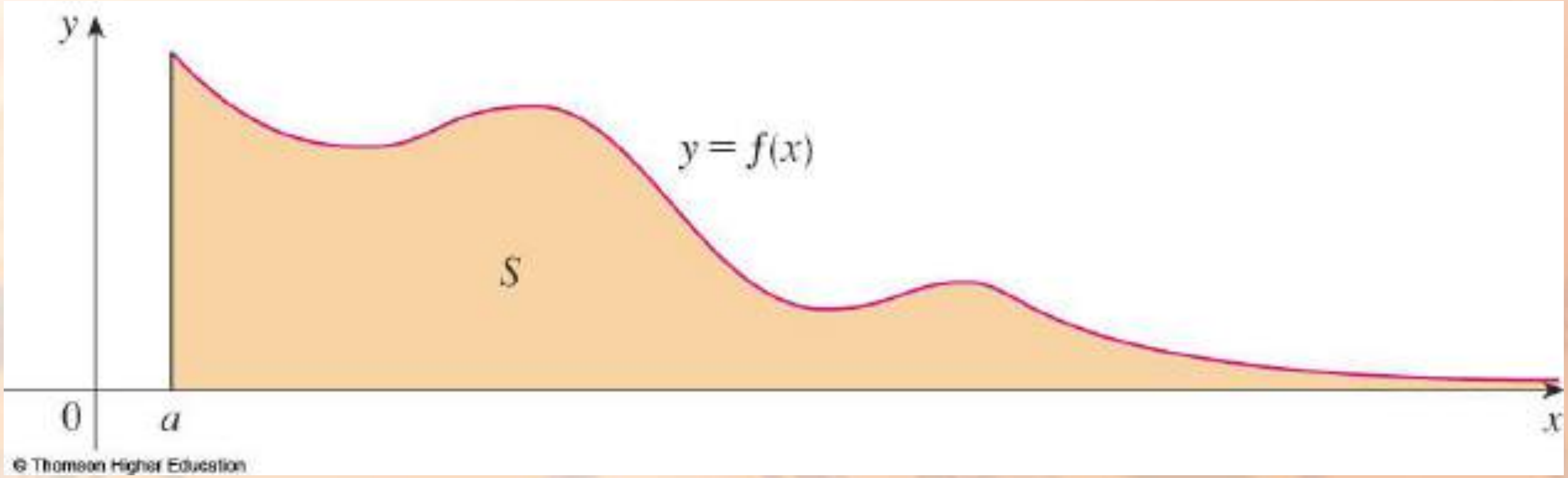
- Then, we define the area of the region $S = \{(x, y) \mid x \geq a, 0 \leq y \leq f(x)\}$ in the figure as:

$$A(S) = \int_a^{\infty} f(x)\,dx$$



IMPROPER INTEGRALS OF TYPE 1

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of *f* from *a* to *t*.



Determine whether the integral

$$\int_1^{\infty} (1/x)\,dx$$

is convergent or divergent.

IMPROPER INTEGRALS OF TYPE 1 **Example 1**

According to Definition 1 a,

we have:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}(\ln t - \ln 1)$$

$$= \lim_{t\to\infty} \ln t = \infty$$

- The limit does not exist as a finite number.
- So, the integral is divergent.

IMPROPER INTEGRALS OF TYPE 1

Let's compare the result of Example 1 with the example at the beginning of the section:

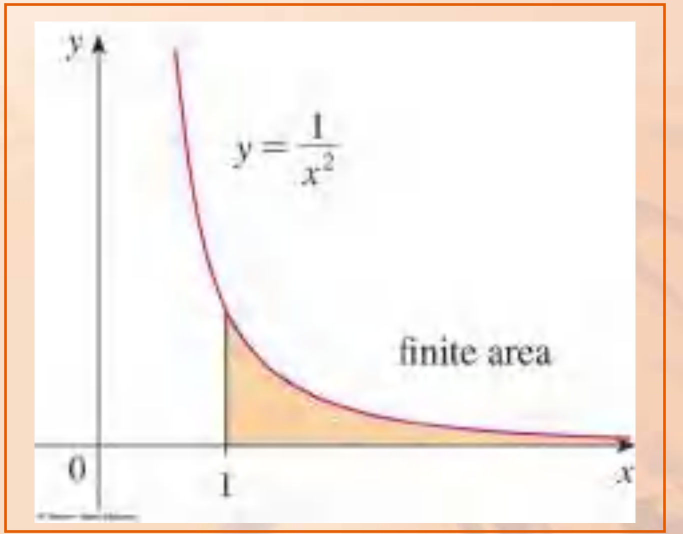
$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

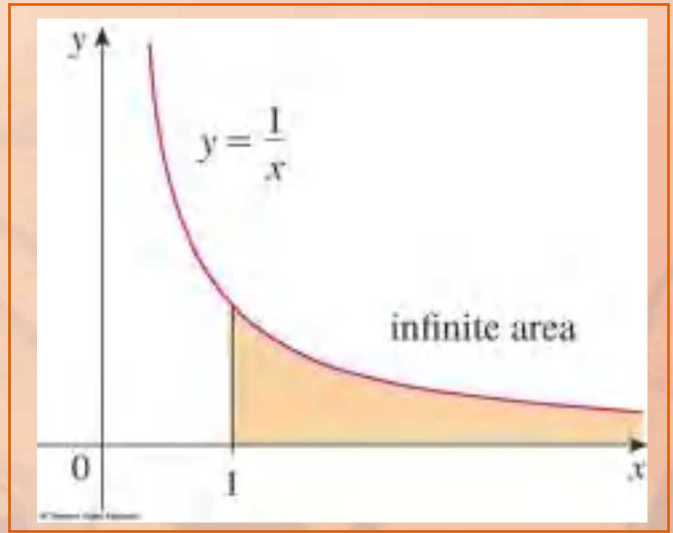
- Geometrically, this means the following.

IMPROPER INTEGRALS OF TYPE 1

The curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$.

However, the region under $y = 1/x^2$ to the right of $x = 1$ has finite area, but the corresponding region under $y = 1/x$ has infinite area.

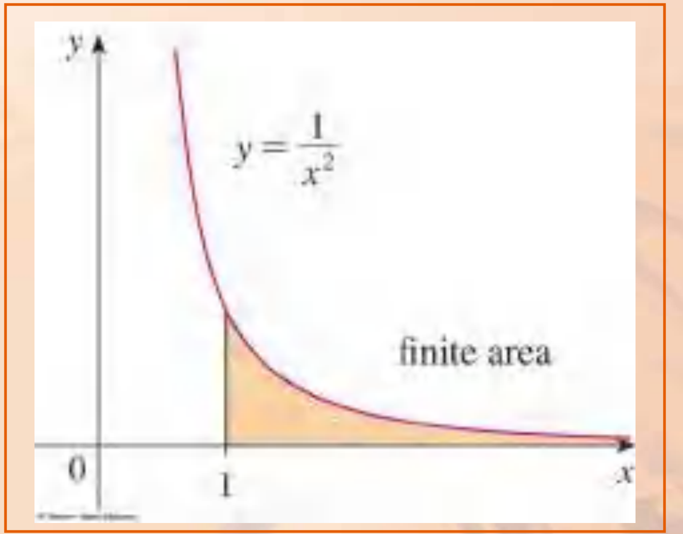


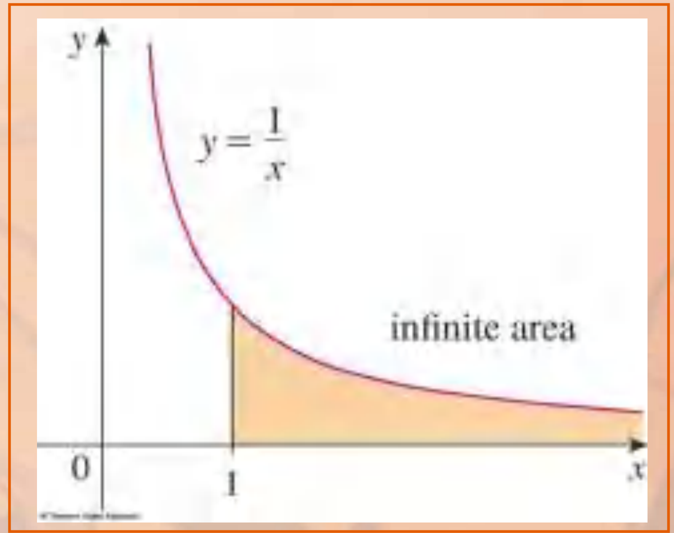


IMPROPER INTEGRALS OF TYPE 1

Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$, but $1/x^2$ approaches faster than $1/x$.

- The values of $1/x$ don't decrease fast enough for its integral to have a finite value.





Example 2

Evaluate $\int_{-\infty}^{0} xe^x \, dx$

- Using Definition 1 b, we have: $\int_{-\infty}^{0} xe^x \, dx = \lim_{t \to -\infty} \int_{t}^{0} xe^x \, dx$

- We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = xe^x\Big]_t^0 - \int_t^0 e^x\,dx$$

$$= -te^t - 1 + e^t$$

- We know that $e^t \to 0$ as $t \to -\infty$, and, by l'Hospital's Rule, we have:

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}}$$

$$= \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} (-e^t)$$

$$= 0$$

- Therefore,

$$\int_{-\infty}^{0} xe^{x}\,dx = \lim_{t\to-\infty}\left(-te^{t} - 1 + e^{t}\right)$$

$$= -0 - 1 + 0$$

$$= -1$$

 Example 3

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

- It's convenient to choose *a* = 0 in Definition 1 c:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

We must now evaluate the integrals on the right side separately—as follows.

IMPROPER INTEGRALS OF TYPE 1 **Example 3**

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty} \tan^{-1} x \Big]_0^t$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0)$$

$$= \lim_{t\to\infty} \tan^{-1} t$$

$$= \frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty} \tan^{-1} x \Big]_t^0$$

$$= \lim_{t\to-\infty} (\tan^{-1} 0 - \tan^{-1} t)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

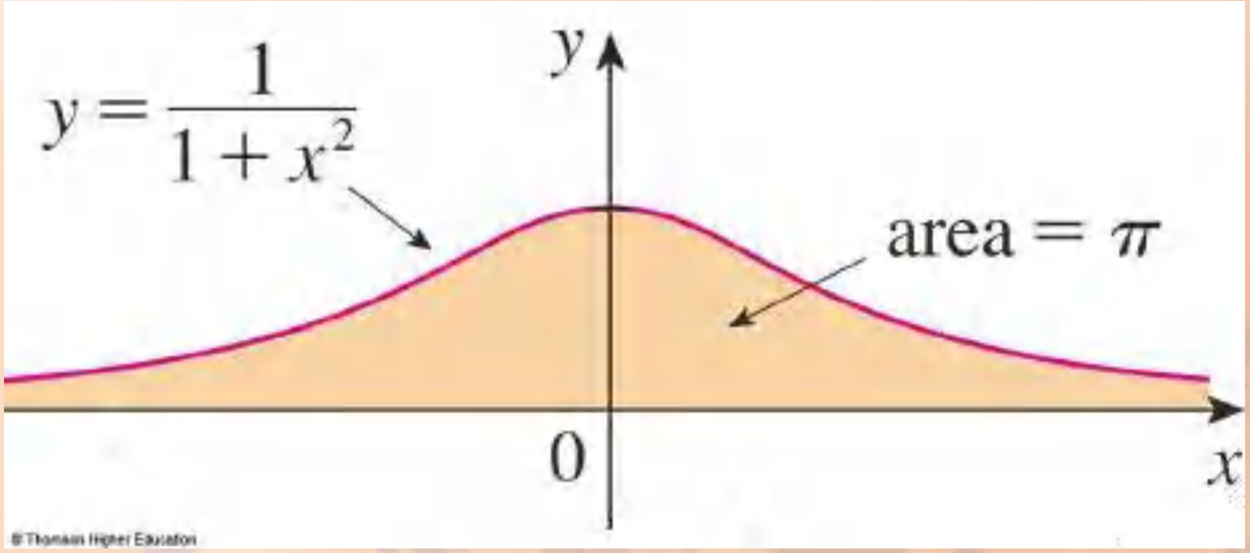
$$= \frac{\pi}{2}$$

Since both these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

IMPROPER INTEGRALS OF TYPE 1 **Example 3**

As $1/(1 + x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x–axis.



IMPROPER INTEGRALS OF TYPE 1 **Example 4**

For what values of *p* is the integral

$\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

- We know from Example 1 that, if $p = 1$, the integral is divergent.
- So, let's assume that $p \neq 1$.

Then,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

IMPROPER INTEGRALS OF TYPE 1 **Example 4**

If $p > 1$, then $p - 1 > 0$.

So, as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.

- Therefore, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{if} \quad p > 1$
- So, the integral converges.

However, if p <1, then p – 1 < 0.

So, $$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \text{ as } t \to \infty$$

- Thus, the integral diverges.

IMPROPER INTEGRALS OF TYPE 1 **Definition 2**

We summarize the result of Example 4 for future reference:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is:}$$

- Convergent if $p > 1$
- Divergent if $p \le 1$

TYPE 2—DISCONTINUOUS INTEGRANDS

Suppose f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b.

DISCONTINUOUS INTEGRANDS

Let *S* be the unbounded region under the graph of *f* and above the *x*-axis between *a* and *b*.

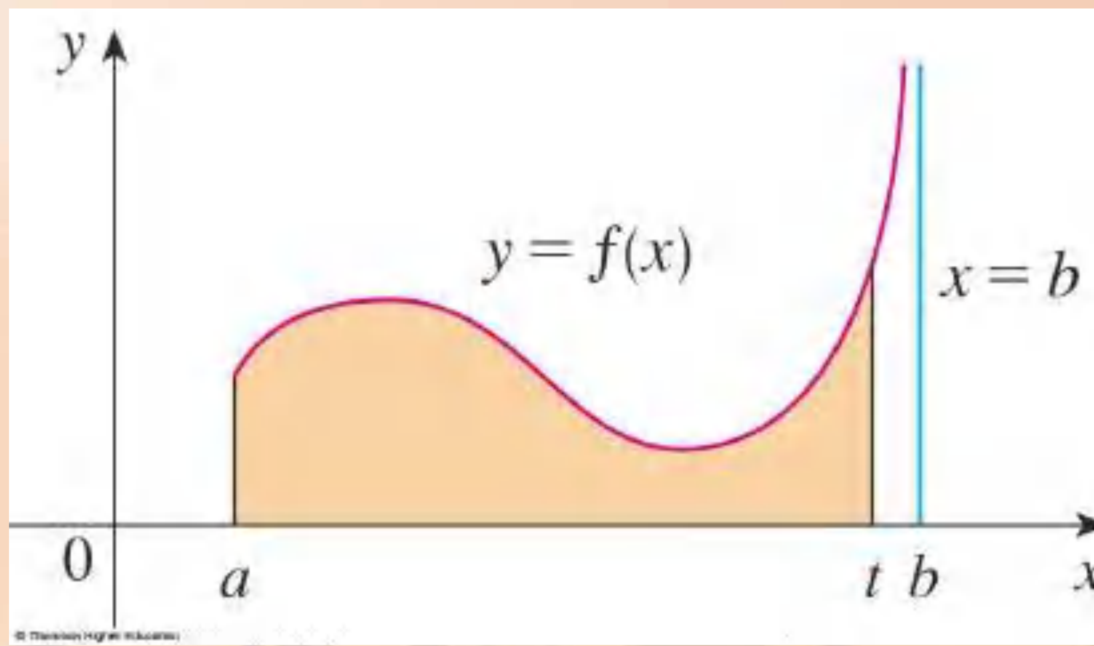
- For Type 1 integrals, the regions extended indefinitely in a horizontal direction.

- Here, the region is infinite in a vertical direction.

DISCONTINUOUS INTEGRANDS

The area of the part of *S* between *a* and *t* (shaded region) is:

$$A(t) = \int_a^t f(x)\,dx$$



DISCONTINUOUS INTEGRANDS

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

DISCONTINUOUS INTEGRANDS

We use the equation to define an improper integral of Type 2 even when *f* is not a positive function—no matter what type of discontinuity *f* has at *b*.

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 a**

If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

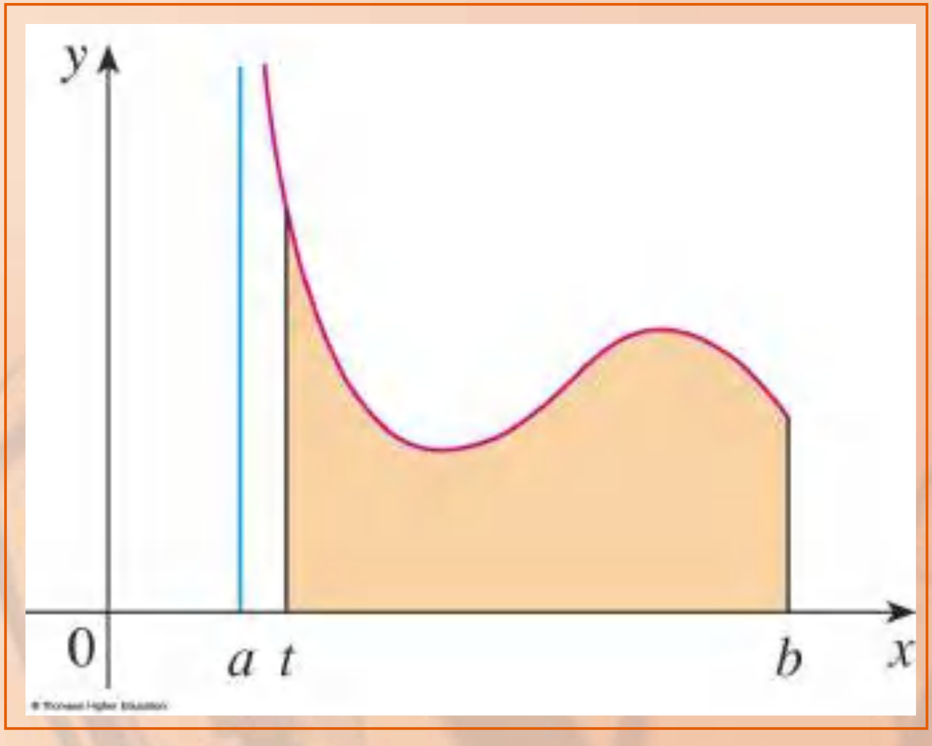
If *f* is continuous on (*a*, *b*] and is discontinuous at *a*, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

Definition 3 b is illustrated for the case where $f(x) \geq 0$ and has vertical asymptotes at a and c, respectively.



IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

The improper integral $\int_a^b f(x)\,dx$

is called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

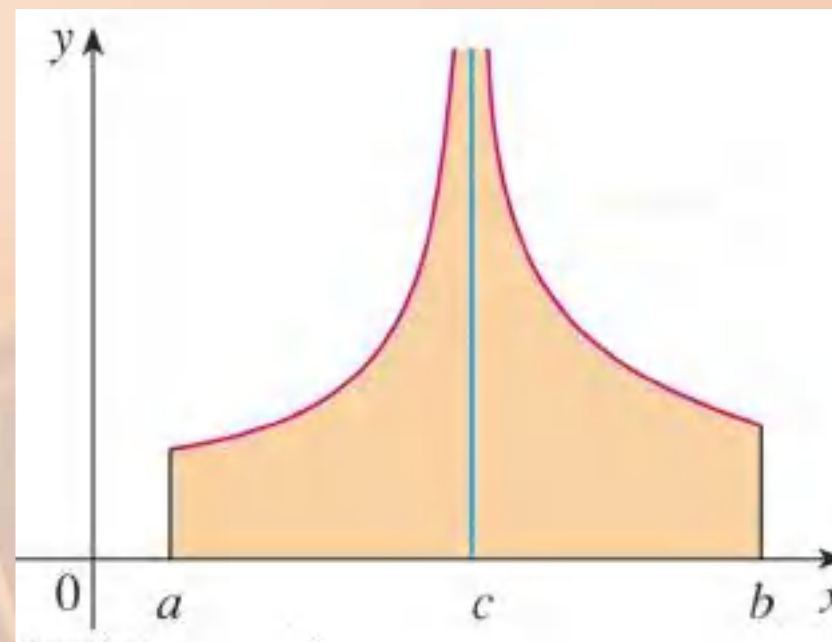
IMPROPER INTEGRAL OF TYPE 2 **Definition 3 c**

If *f* has a discontinuity at *c*, where *a* < *c* < *b*, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 c**

Definition 3 c is illustrated for the case where $f(x) \geq 0$ and has vertical asymptotes at a and c, respectively.



IMPROPER INTEGRALS OF TYPE 2 **Example 5**

Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

- First, we note that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$.

IMPROPER INTEGRALS OF TYPE 2 **Example 5**

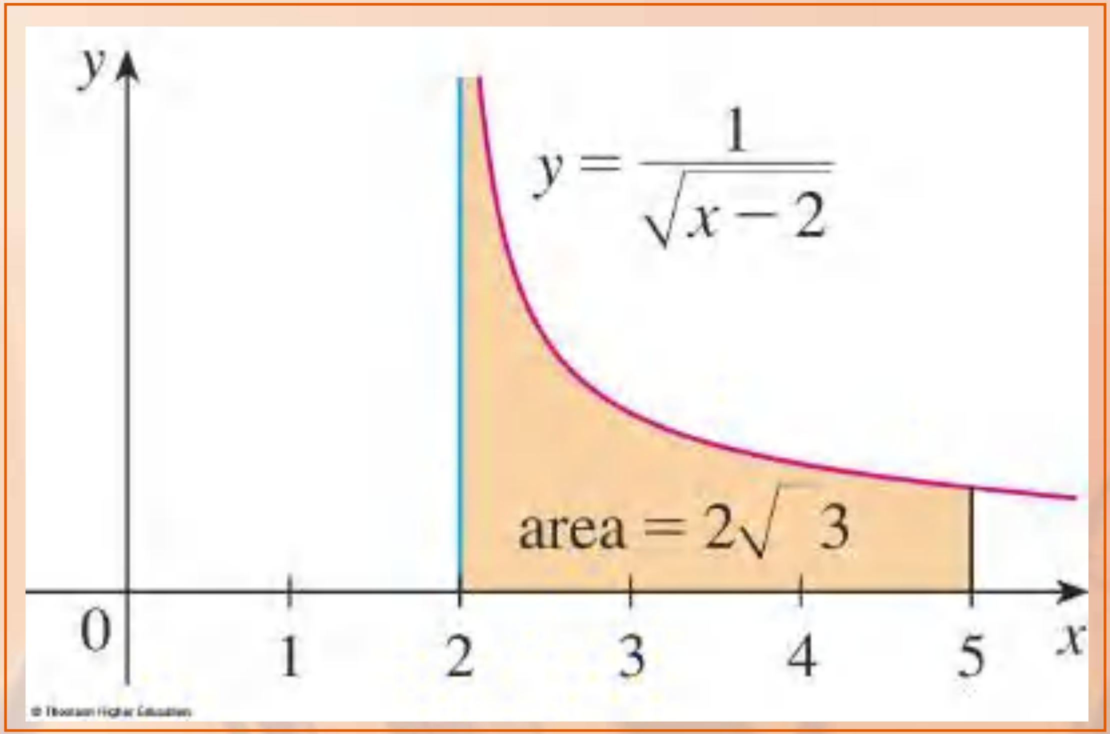
- The infinite discontinuity occurs at the left end-point of [2, 5].
- So, we use Definition 3 b:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5 \\ &= \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2}) \\ &= 2\sqrt{3}\end{aligned}$$

- Thus, the given improper integral is convergent.

IMPROPER INTEGRALS OF TYPE 2 **Example 5**

- Since the integrand is positive, we can interpret the value of the integral as the area of the shaded region here.



Determine whether $\int_0^{\pi/2} \sec x \, dx$ converges or diverges.

- Note that the given integral is improper because: $$\lim_{x \to (\pi/2)^-} \sec x = \infty$$

IMPROPER INTEGRALS OF TYPE 2 **Example 6**

- Using Definition 2 a, we have:

$$\int_0^{\pi/2} \sec x\, dx = \lim_{x \to (\pi/2)^-} \int_0^t \sec x\, dx$$

$$= \lim_{x \to (\pi/2)^-} \ln\left|\sec x + \tan x\right| \Big]_0^t$$

$$= \lim_{x \to (\pi/2)^-} \left[\ln(\sec t + \tan t) - \ln 1\right] = \infty$$

- This is because sec $t \to \infty$ and tan $t \to \infty$ as $t \to (\pi/2)^-$.

- Thus, the given improper integral is divergent.

Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

- Observe that the line $x = 1$ is a vertical asymptote of the integrand.

IMPROPER INTEGRALS OF TYPE 2 **Example 7**

- As it occurs in the middle of the interval [0, 3], we must use Definition 3 c with c = 1:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} |x-1| \Big]_0^t$$

$$= \lim_{t\to 1^-} (\ln|t-1| - \ln|-1|)$$

$$= \lim_{t\to 1^-} \ln(1-t) = -\infty$$

- This is because $1 - t \to 0^+$ as $t \to 1^-$.

IMPROPER INTEGRALS OF TYPE 2 **Example 7**

Thus, $\int_0^1 dx/(x-1)$ is divergent.

This implies that $\int_0^3 dx/(x-1)$ is divergent.

- We do not need to evaluate $\int_1^3 dx/(x-1)$.

WARNING

Then, we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

- This is wrong because the integral is improper and must be calculated in terms of limits.

WARNING

From now, whenever you meet the symbol $\int_a^b f(x)\,dx$, you must decide, by looking at the function *f* on [*a*, *b*], whether it is either:

- An ordinary definite integral
- An improper integral

Evaluate $\int_0^1 \ln x \, dx$

- We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x \to 0^+} \ln x = -\infty$.

- Thus, the given integral is improper, and we have:

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

- Now, we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x\,dx = x\ln x\Big]_t^1 - \int_t^1 dx$$

$$= 1\ln 1 - t\ln t - (1-t)$$

$$= -t\ln t - 1 + t$$

IMPROPER INTEGRALS OF TYPE 2 **Example 8**

- To find the limit of the first term, we use l'Hospital's Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t}$$

$$= \lim_{t\to 0^+} \frac{1/t}{-1/t^2}$$

$$= \lim_{t\to 0^+} (-t)$$

$$= 0$$

IMPROPER INTEGRALS OF TYPE 2 **Example 8**

- Therefore,

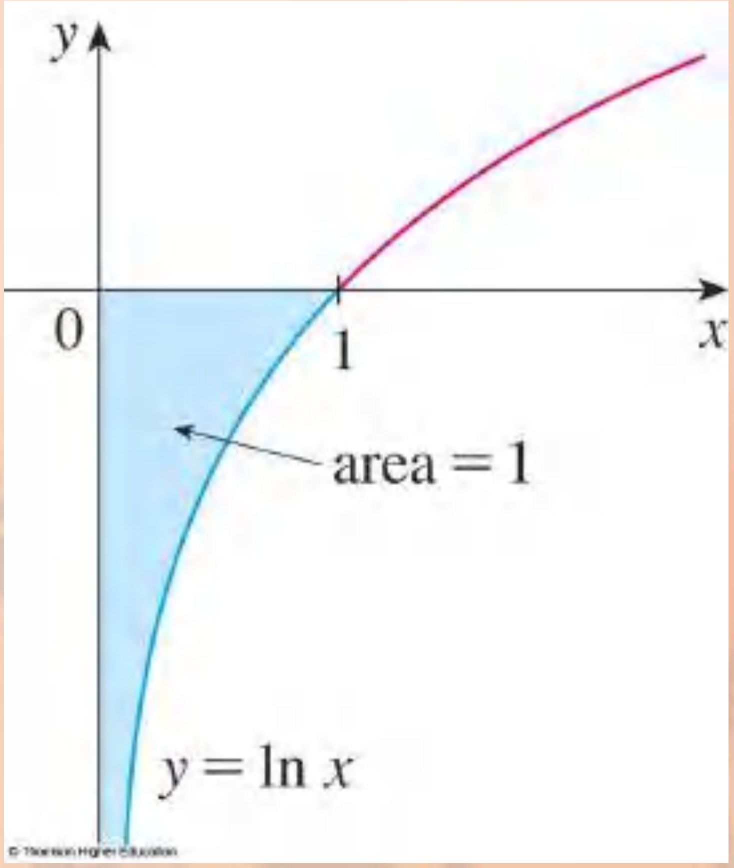
$$\int_0^1 \ln x\, dx = \lim_{t \to 0^+} (-t \ln t - 1 + t)$$

$$= -0 - 1 + 0$$

$$= -1$$

The geometric interpretation of the result is shown.

- The area of the shaded region above $y = \ln x$ and below the x-axis is 1.



A COMPARISON TEST FOR IMPROPER INTEGRALS

Sometimes, it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent.

- In such cases, the following theorem is useful.
- Although we state it for Type 1 integrals,
  a similar theorem is true for Type 2 integrals.

COMPARISON THEOREM

Suppose *f* and *g* are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

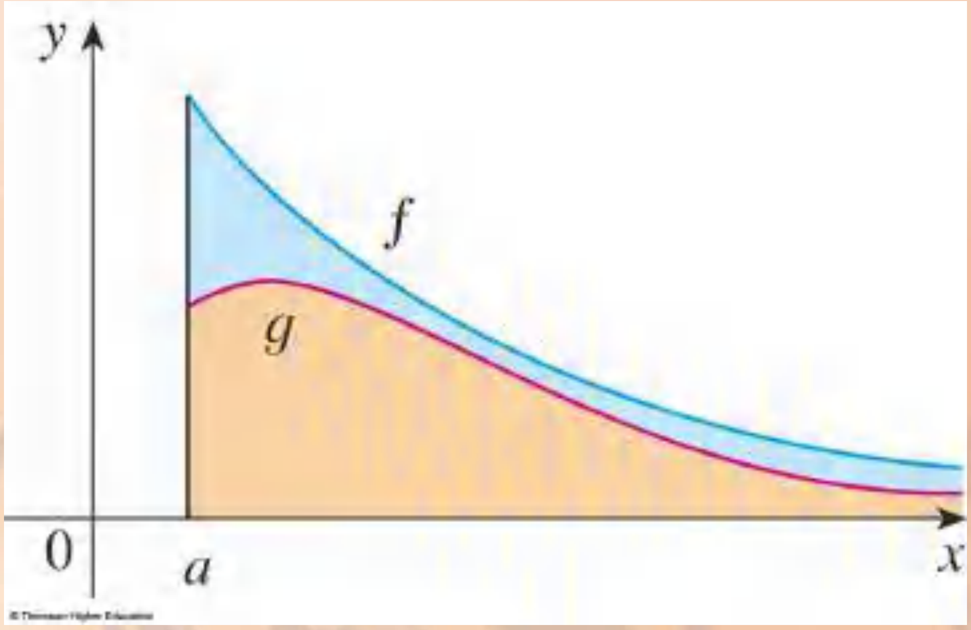
a. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

b. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

COMPARISON THEOREM

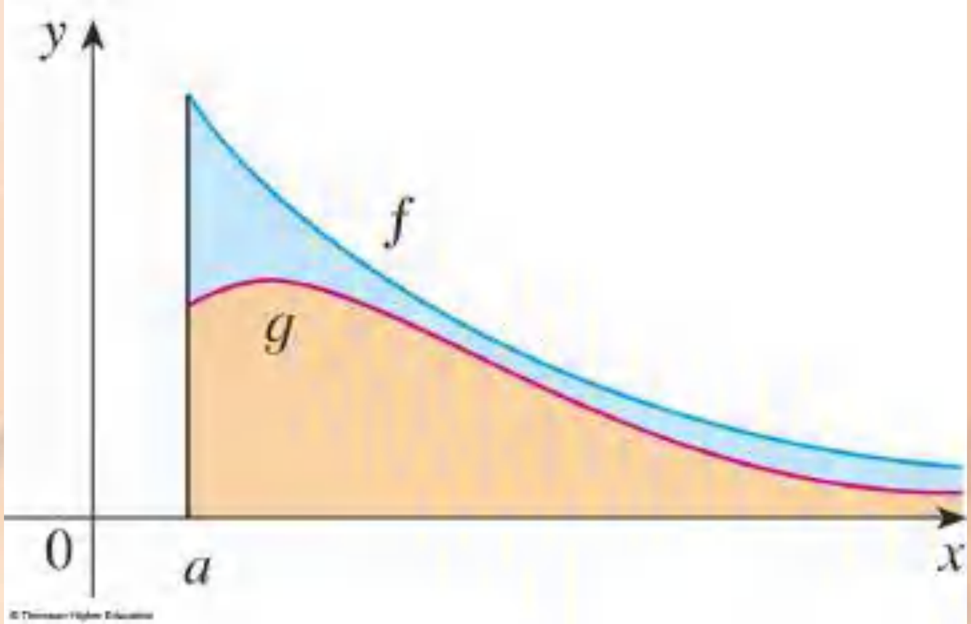
We omit the proof of the theorem.

However, the figure makes it seem plausible.



COMPARISON THEOREM

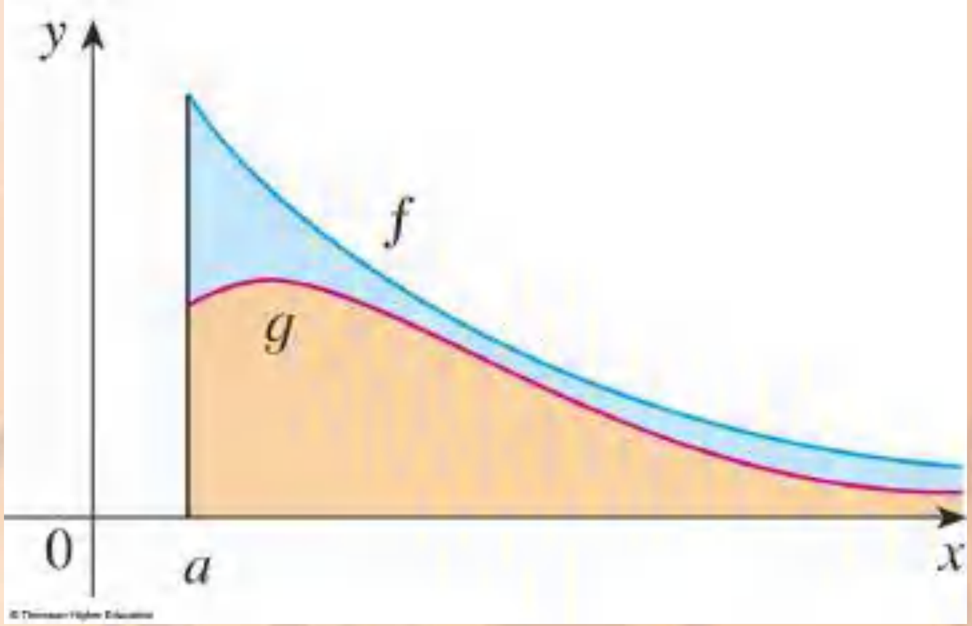
If the area under the top curve $y = f(x)$ is finite, so is the area under the bottom curve $y = g(x)$.



COMPARISON THEOREM

If the area under $y = g(x)$ is infinite,

so is the area under $y = f(x)$.



COMPARISON THEOREM

Note that the reverse is not necessarily true:

- If $\int_a^\infty g(x)\,dx$ is convergent, $\int_a^\infty f(x)\,dx$ may or may not be convergent.
- If $\int_a^\infty f(x)\,dx$ is divergent, $\int_a^\infty g(x)\,dx$ may or may not be divergent.

COMPARISON THEOREM

Example 9

Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

- We can't evaluate the integral directly.
- The antiderivative of e^{-x^2} is not an elementary function (as explained in Section 7.5).

We write:

$$\int_0^{\infty} e^{-x^2}\,dx = \int_0^{1} e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$$

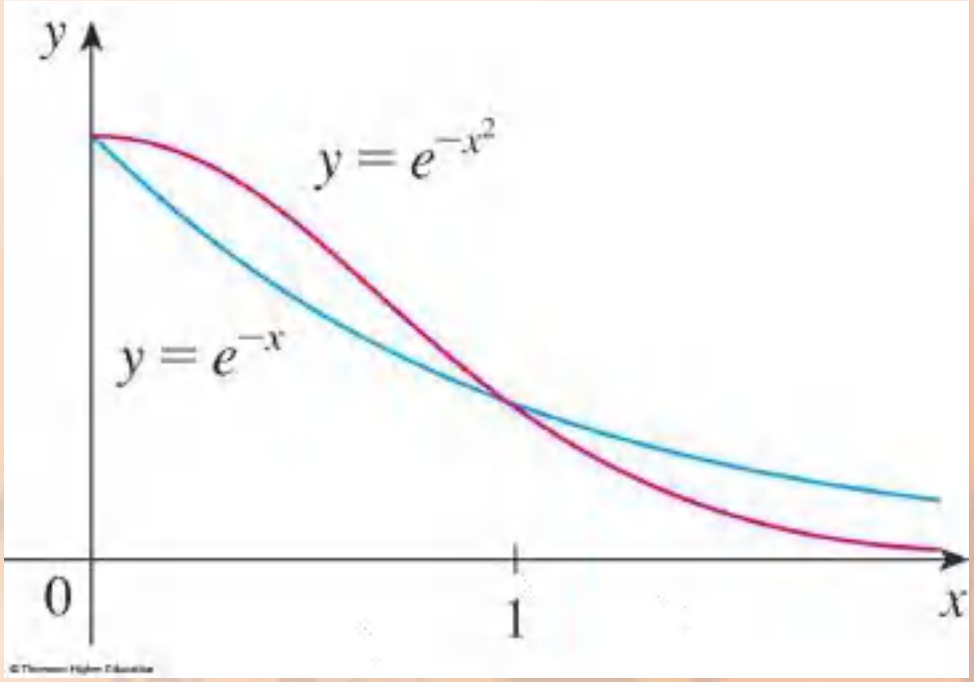
- We observe that the first integral on the right-hand side is just an ordinary definite integral.

COMPARISON THEOREM

Example 9

- In the second integral, we use the fact that, for $x \geq 1$, we have $x^2 \geq x$.

- So, $-x^2 \leq -x$ and, therefore, $e^{-x^2} \leq e^{-x}$.



The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty}(e^{-1} - e^{-t})$$

$$= e^{-1}$$

COMPARISON THEOREM

Example 9

Thus, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the theorem, we see that $\int_1^{\infty} e^{-x^2}\,dx$ is convergent.

- It follows that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

COMPARISON THEOREM

In Example 9, we showed that $\int_0^\infty e^{-x^2}\,dx$ is convergent without computing its value.

- In Exercise 70, we indicate how to show that its value is approximately 0.8862
- In probability theory, it is important to know the exact value of this improper integral.
- Using the methods of multivariable calculus, it can be shown that the exact value is $\sqrt{\pi}/2$.

COMPARISON THEOREM

The table illustrates the definition of an improper integral by showing how the (computer- generated) values of $\int_0^t e^{-x^2}\,dx$ approach $\sqrt{\pi}/2$ as t becomes large.

- In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

TABLE 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

Example 10

The integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem since $\frac{1+e^{-x}}{x} > \frac{1}{x}$

$\int_1^\infty (1/x)\,dx$ is divergent by Example 1 or by Definition 2 with $p = 1$.