POTENTIAL FLOW

Stream Function & Velocity Potential

- Stream lines/ Stream Function (Ψ)
  - Concept
  - Relevant Formulas
  - Examples
  - Rotation, vorticity
- Velocity Potential(ϕ)
  - Concept
  - Relevant Formulas
  - Examples
  - Relationship between stream function and velocity potential
  - Complex velocity potential

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Stream Lines

- Consider 2D incompressible flow
- Continuity Eqn

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho V_x)+\frac{\partial}{\partial y}(\rho V_y)+\frac{\partial}{\partial z}(\rho V_z)=0$$

$$\frac{\partial}{\partial x}(V_x)+\frac{\partial}{\partial y}(V_y)=0 \qquad V_y=\int\left(-\frac{\partial V_x}{\partial x}\right)dy$$

- V_x and V_y are related
- Can you write a common function for both?

Stream Function

- Assume

$$V_x = \frac{\partial \psi}{\partial y}$$

- Then

$$V_y = \int \left(-\frac{\partial V_x}{\partial x} \right) dy = \int \left(-\frac{\partial^2 \psi}{\partial x \partial y} \right) dy$$

$$= \int \left(-\frac{\partial^2 \psi}{\partial y \partial x} \right) dy = \left(-\frac{\partial \psi}{\partial x} \right)$$

- Instead of two functions, V_x and V_y, we need to solve for only one function ψ – Stream Function
- Order of differential eqn increased by one

Stream Function

- What does Stream Function ψ mean?

- Equation for streamlines in 2D are given by

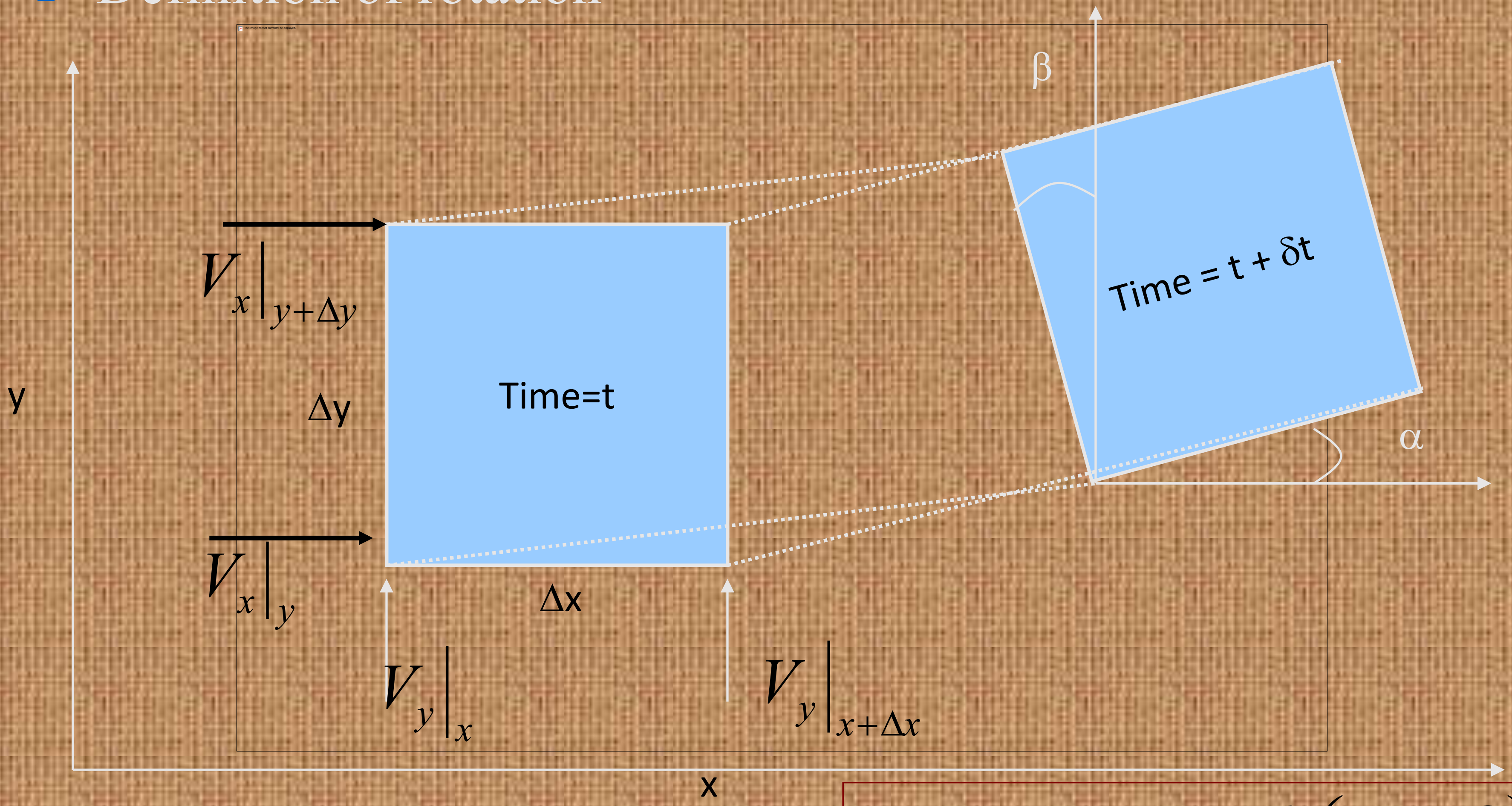
  ψ = constant

- Streamlines may exist in 3D also, but stream function does not
  - Why? (When we work with velocity potential, we may get a perspective)
  - In 3D, streamlines follow the equation

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$

Rotation

- Definition of rotation



Assume $V_y|_x < V_y|_{x+\Delta x}$

and $V_x|_y > V_x|_{y+\Delta y}$

$$\text{ROTATION} = \omega_z = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right)$$

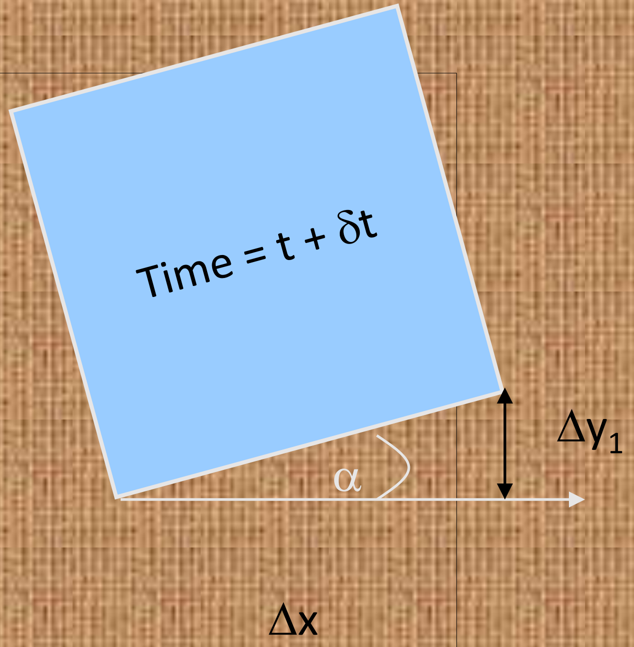
Rotation

- To Calculate Rotation

$$\tan\alpha = \frac{\Delta y_1}{\Delta x}$$

$$\Delta y_1 = \left(V_y\Big|_{x+\Delta x}\Delta t\right) - \left(V_y\Big|_{x}\Delta t\right)$$

$$\alpha = \arctan\frac{\left(V_y\Big|_{x+\Delta x} - V_y\Big|_{x}\right)\Delta t}{\Delta x}$$



- Similarly

$$\beta = \arctan\frac{-\left(V_x\Big|_{y+\Delta y} - V_x\Big|_{y}\right)\Delta t}{\Delta y}$$

Rotation

- To Calculate Rotation

$$\text{ROTATION} = \omega_z = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\lim_{\Delta t\to 0}\left(\frac{(\alpha+\beta)\big|_{t+\Delta t} - (\alpha+\beta)\big|_{t}}{\Delta t}\right)$$

$$= \left(\frac{1}{2}\right)\lim_{\substack{\Delta t\to 0\\ \Delta x\to 0\\ \Delta y\to 0}} \frac{\arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_{x}\right)\Delta t}{\Delta x}\right\}}{\Delta t} \quad \alpha \quad \left(\frac{1}{2}\right)\lim_{\substack{\Delta t\to 0\\ \Delta x\to 0\\ \Delta y\to 0}} \frac{\arctan\left\{\frac{\left(V_x\big|_{y+\Delta y} - V_x\big|_{y}\right)\Delta t}{\Delta y}\right\}}{\Delta t} \quad \beta$$

- For very small time and very small element, Δx, Δy and Δt are close to zero

Rotation

- To Calculate Rotation
- For very small θ, (i.e.θ ~ 0)

$$\sin\theta \cong \theta \qquad \cos\theta \cong 1 \qquad \Rightarrow \tan\theta \cong \theta$$

$$\therefore \quad \arctan\theta \cong \theta$$

$$\therefore \quad \arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\} \cong \frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}$$

$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\}}{\Delta t} = \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\}}{\Delta t}$$

Rotation

- To Calculate Rotation

$$\lim_{\Delta x \to 0} \frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x \right)}{\Delta x} = \frac{\partial V_y}{\partial x}$$

$$\omega_z = \left(\frac{1}{2}\right) \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\arctan\left\{ \frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x \right)\Delta t}{\Delta x} \right\}}{\Delta t} - \left(\frac{1}{2}\right) \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\arctan\left\{ \frac{\left(V_x\big|_{y+\Delta y} - V_x\big|_y \right)\Delta t}{\Delta y} \right\}}{\Delta t}$$

Simplifies to

$$\omega_z = \left(\frac{1}{2}\right)\left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\}$$

$$\omega_z = \left(\frac{1}{2}\right)\left(\nabla \times \dot{V}\right)$$

Rotation in terms of Stream Function

- To write rotation in terms of stream functions

$$V_x = \frac{\partial \psi}{\partial y} \qquad V_y = \left(-\frac{\partial \psi}{\partial x}\right)$$

$$\omega_z = \left(\frac{1}{2}\right)\left\{\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right\} = \left(\frac{1}{2}\right)\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$= -\left(\frac{1}{2}\right)\nabla^2 \psi$$

- That is

$$\nabla^2 \psi + 2\omega_z = 0$$

- For irrotational flow (ω_z=0)

$$\nabla^2 \psi = 0$$

Rotation and Potential

- For irrotational flow ($\omega_z = 0$)

$$\omega_z = \frac{1}{2}\left(\nabla \times \vec{V}\right) = 0$$

$$\nabla \times \bar{V} = 0$$

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 0$$

- This equation is "similar" to continuity equation
- V_x and V_y are related
- Can we find a common function to relate both V_x and V_y ?

Velocity Potential

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 0$$

- Assume

$$V_x = \frac{\partial \phi}{\partial x}$$

- Then $$\frac{\partial V_y}{\partial x} = \frac{\partial V_x}{\partial y} \quad = \frac{\partial^2 \phi}{\partial y \partial x} \quad = \frac{\partial^2 \phi}{\partial x \partial y}$$

$$V_y = \frac{\partial \phi}{\partial y}$$

- In 3D, similarly it can be shown that

$$V_z = \frac{\partial \phi}{\partial z}$$

- ϕ is the velocity potential

Velocity Potential vs Stream Function

	Stream Function (ψ)	Velocity Potential (ϕ)
Exists for	only 2D flow	all flows
	viscous or non-viscous flows	Irrotational (i.e. Inviscid or zero viscosity) flow
	Incompressible flow (steady or unsteady)	Incompressible flow (steady or unsteady state)
	compressible flow (steady state only)	compressible flow (steady or unsteady state)

- In 2D inviscid flow (incompressible flow OR steady state compressible flow), both functions exist
- What is the relationship between them?

Stream Function- Physical meaning

- Statement: In 2D (viscous or inviscid) flow (incompressible flow OR steady state compressible flow), ψ = constant represents the streamline.
- Proof

- If ψ = constant, then dψ=0

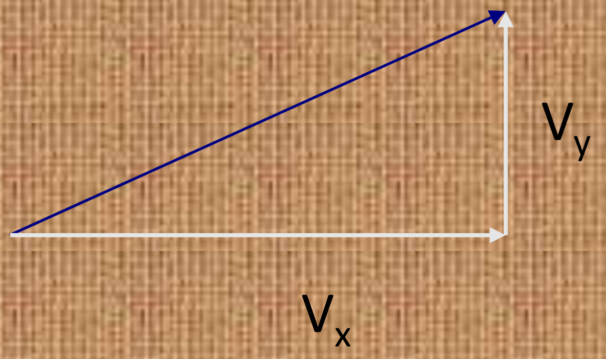
$$d\psi = \left(\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \psi}{\partial y}\right)dy$$

$$= \left(-V_y\right)dx + \left(V_x\right)dy$$

$$= 0$$

- If ψ = constant, then

$$\frac{dy}{dx} = \frac{V_y}{V_x}$$



VISCOUS FLOW

i) Viscous Sublayer (VSL)

The thickness of the VSL (δ the lower case Greek letter delta) is known from experiments to be related to the kinematic viscosity and the shear velocity of the flow by:

$$\delta = \frac{12\nu}{U_*} \quad \text{(units in metres)}$$

It ranges from a fraction of a millimetre to several millimetres thick.

The thickness of the VSL is particularly important in comparison to size of grains (d) on the bed (we'll see later that the forces that act on the grains vary with this relationship).

The *Boundary Reynolds' Number* (**R**$_*$) is used to determine the relationship between δ and d:

$$\mathbf{R}_* = \frac{U_* d}{\nu}$$

A key question is "at what value of **R**$_*$ is the diameter of the grains on the bed equal to the thickness of the VSL?"

Given that: $$\delta = \frac{12\nu}{U_*}$$ The condition exists when δ = d.

Substituting: $$\mathbf{R}_* = \frac{\cancel{U_*}}{\cancel{\nu}} \times \frac{12\cancel{\nu}}{\cancel{U_*}} = 12$$

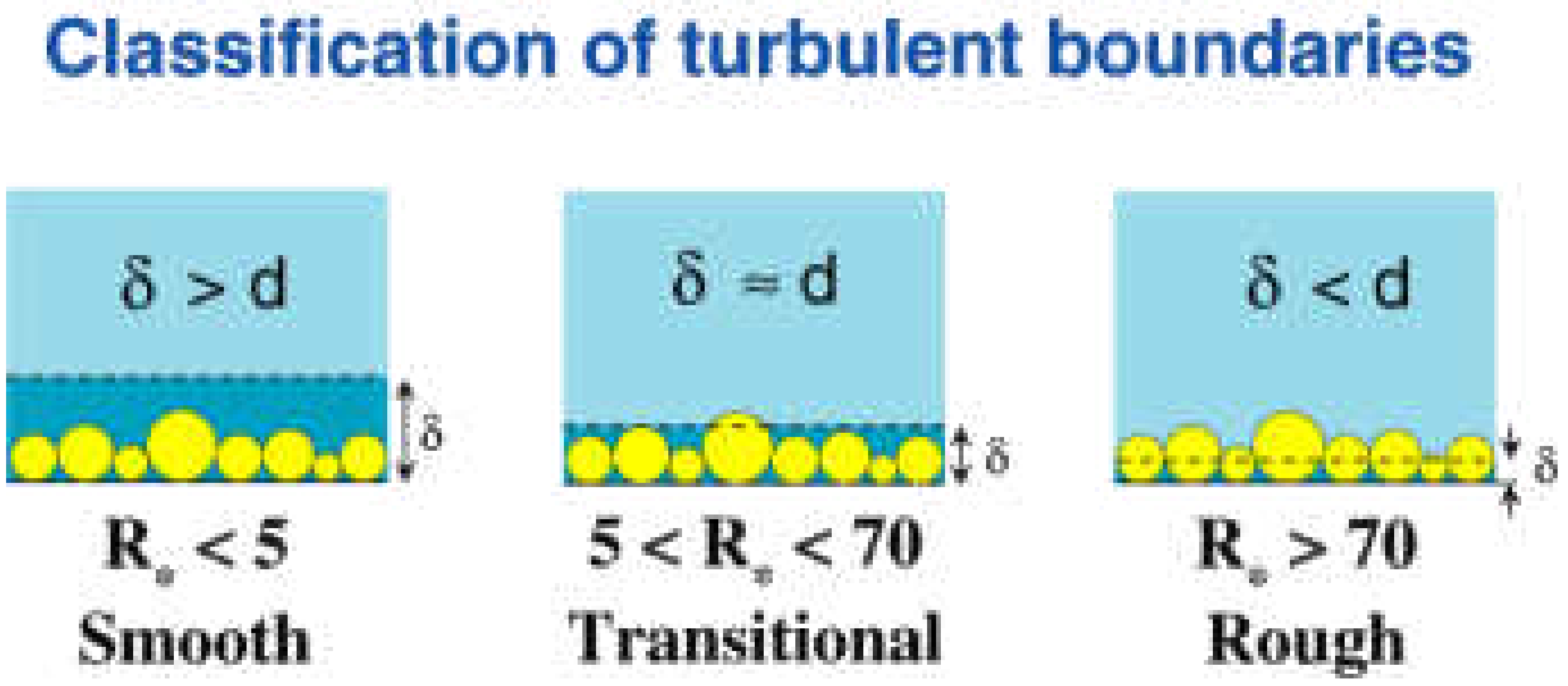
$R_* < 12$ $\delta > d$

$R_* = 12$ $\delta = d$

$R_* > 12$ $\delta < d$

Turbulent boundaries are classified on the basis of the relationship between thickness of the VSL and the size of the bed material.

Given that there is normally a range in grain size on the boundary, the following shows the classification:

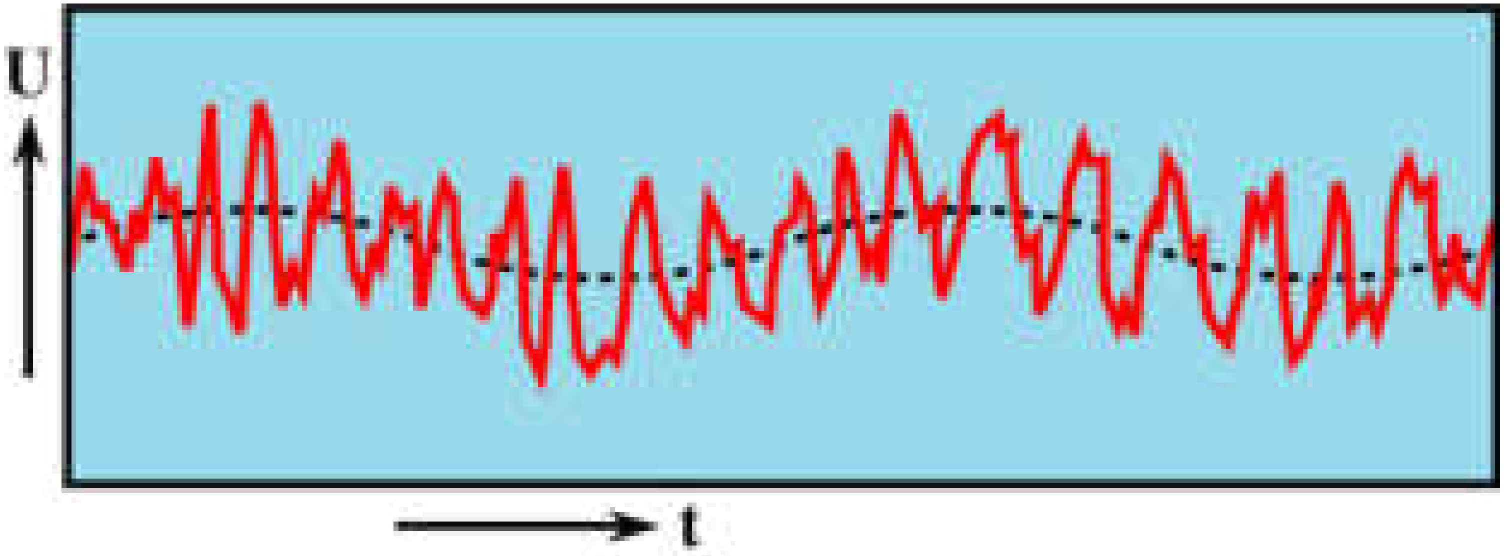


IV. Organized structure of turbulent flows

We characterized turbulent flows as being of a “chaotic” nature marked by random fluid motion.

More accurately, turbulence consists of organized structures of various scale with randomness likely superimposed.

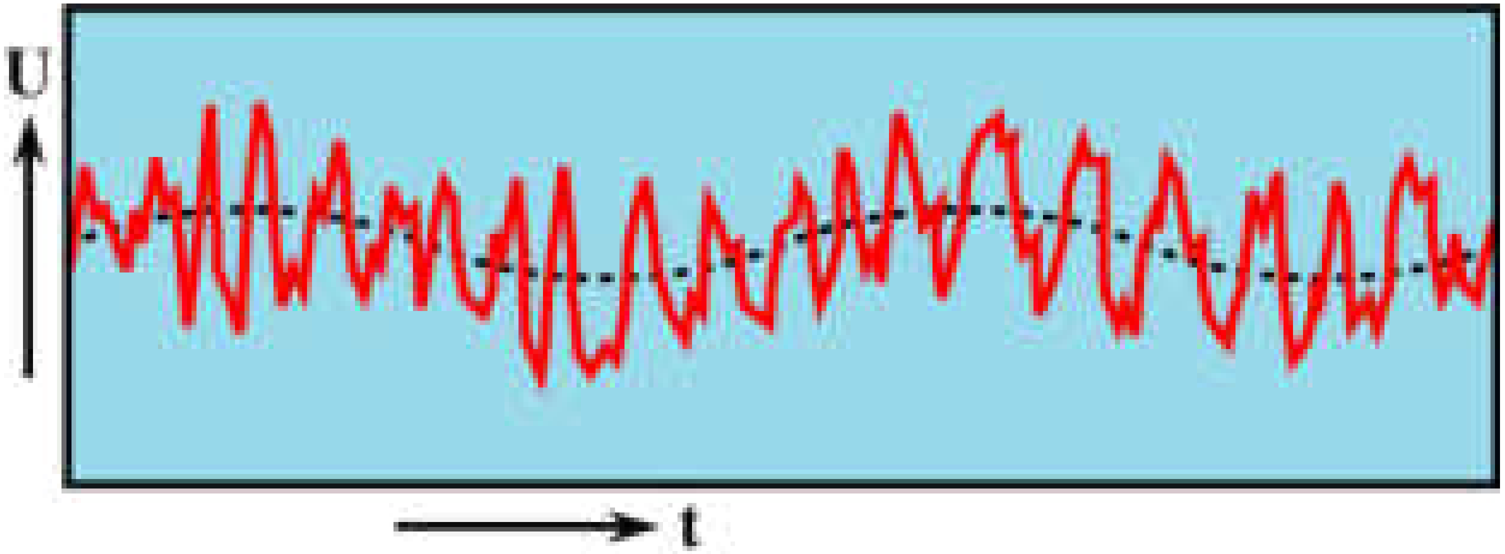
The following illustration shows a hypothetical record of changing flow velocity at a point in a flow.



Note that there are short duration, relatively large magnitude fluctuations that are superimposed on a longer duration, lower magnitude, regular variation in velocity.

Such a pattern of velocity fluctuations is due to large and small scale organized structures.

Note that a similar pattern of variation would be apparent if boundary shear stress were plotted instead of velocity.



Note on boundary shear stress, erosion and deposition

At the boundary of a turbulent flow the **average** boundary shear stress (τ_o) can be determined using the same relationship as for a laminar flow.

In the viscous sublayer viscous shear predominates so that the same relationship exists:

$$\tau_o = \rho g D \sin\theta$$

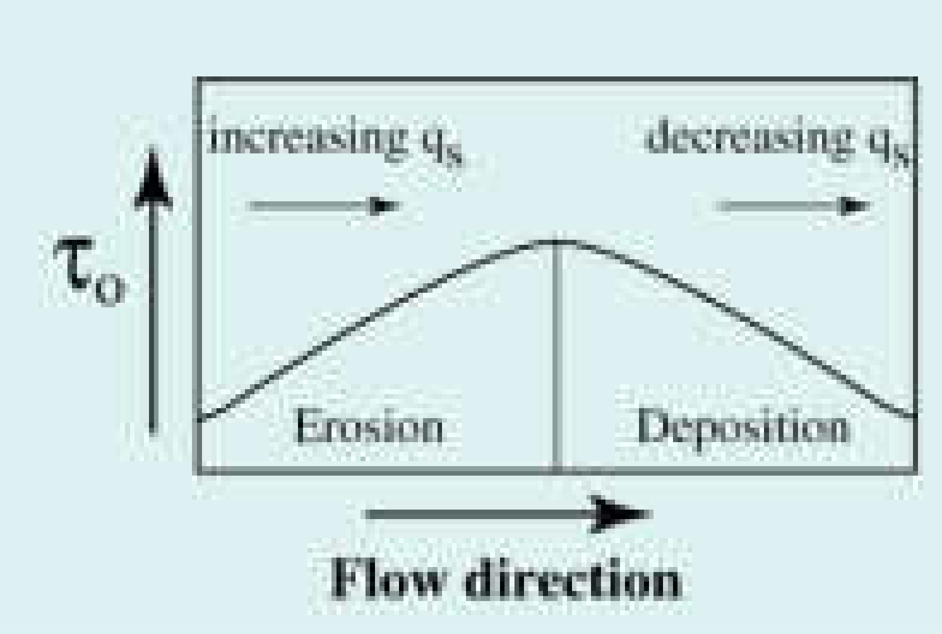
This applies to steady, uniform turbulent flows.

Boundary shear stress governs the power of the current to move sediment; specifically, erosion and deposition depend on the change in boundary shear stress in the downstream direction.

In general, sediment transport rate (q_s; the amount of sediment that is moved by a current) increases with increasing boundary shear stress.

When τ_o increases downstream, so does the sediment transport rate; this leads to erosion of the bed (providing that τ_o is sufficient to move the sediment).

When τ_o decreases downstream, so does the sediment transport rate; this leads to deposition of sediment on the bed



Variation in τ_o along the flow due to turbulence leads to a pattern of erosion and deposition on the bed of a mobile sediment.

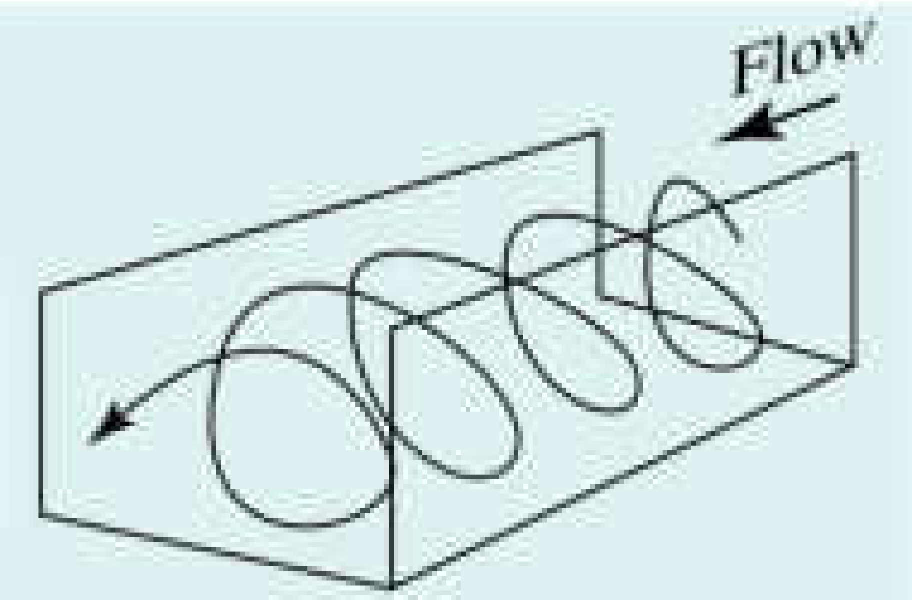
a) Large scale structures of the outer layer

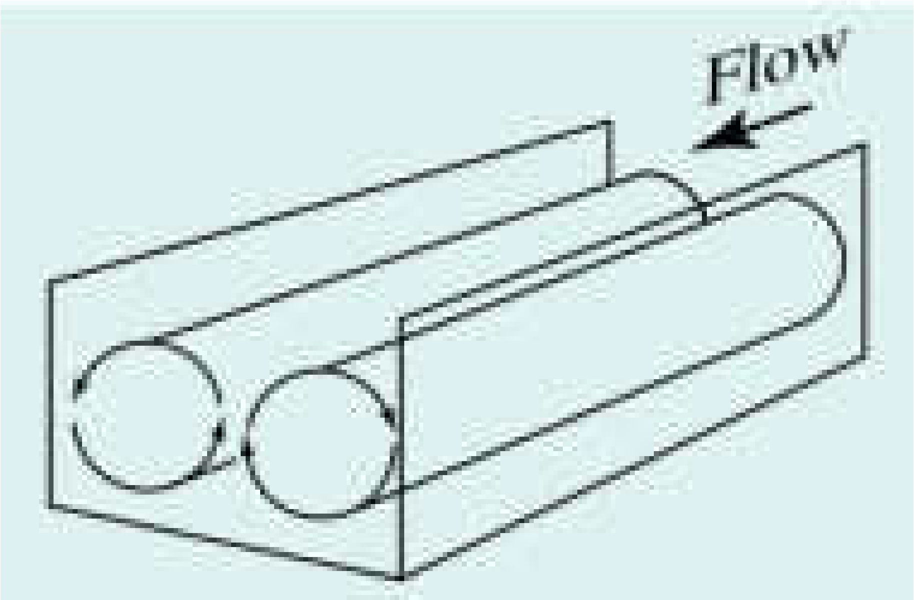
Rotational structures in the outer layer of a turbulent flow.

i) Secondary flows.

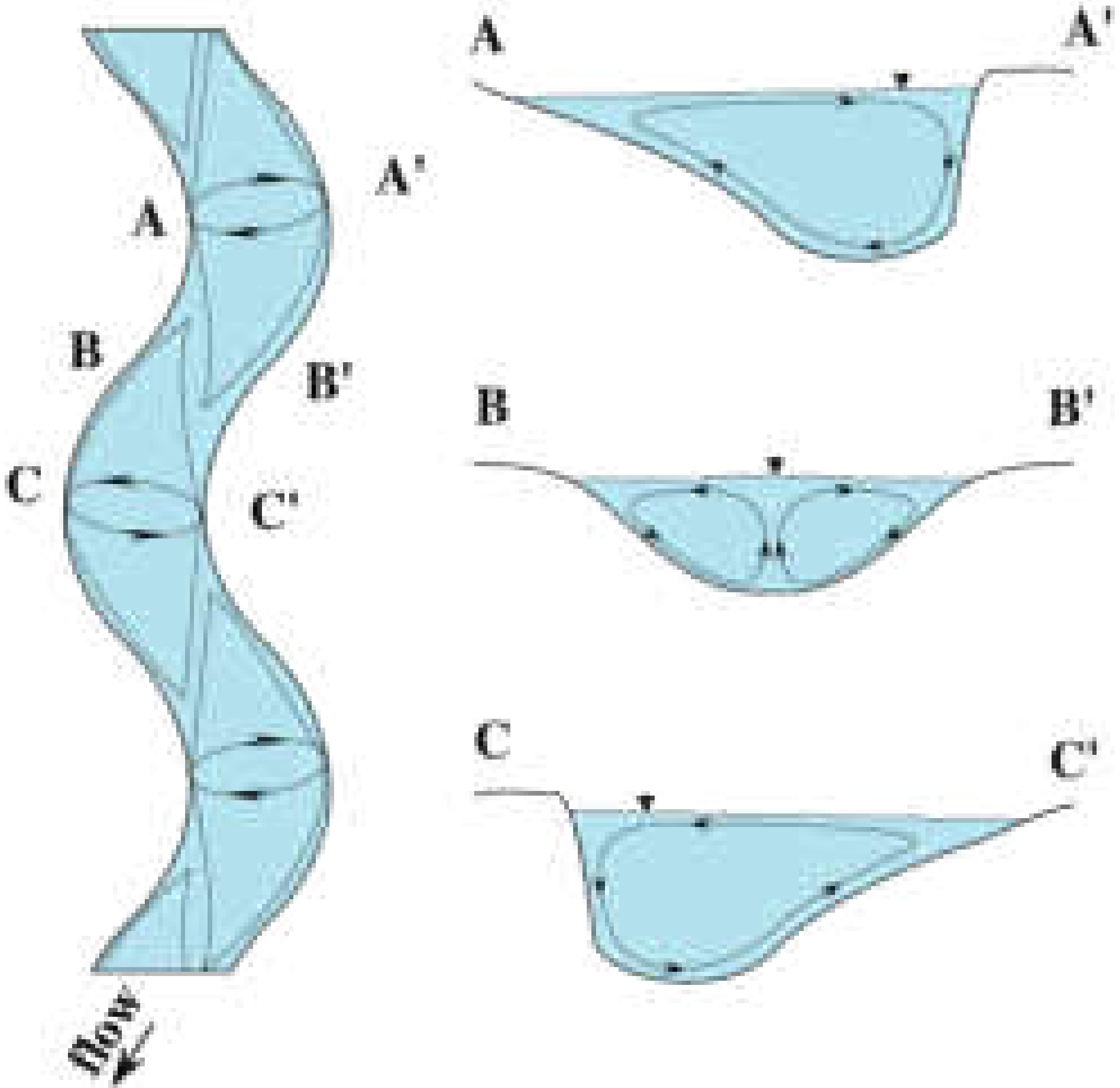
Involves a rotating component of the motion of fluid about an axis that is parallel to the mean flow direction.

Commonly there are two or more such rotating structures extending parallel to each other.





In meandering channels, characterized by a sinusoidal channel form, counter-rotating spiral cells alternate from side to side along the channel.

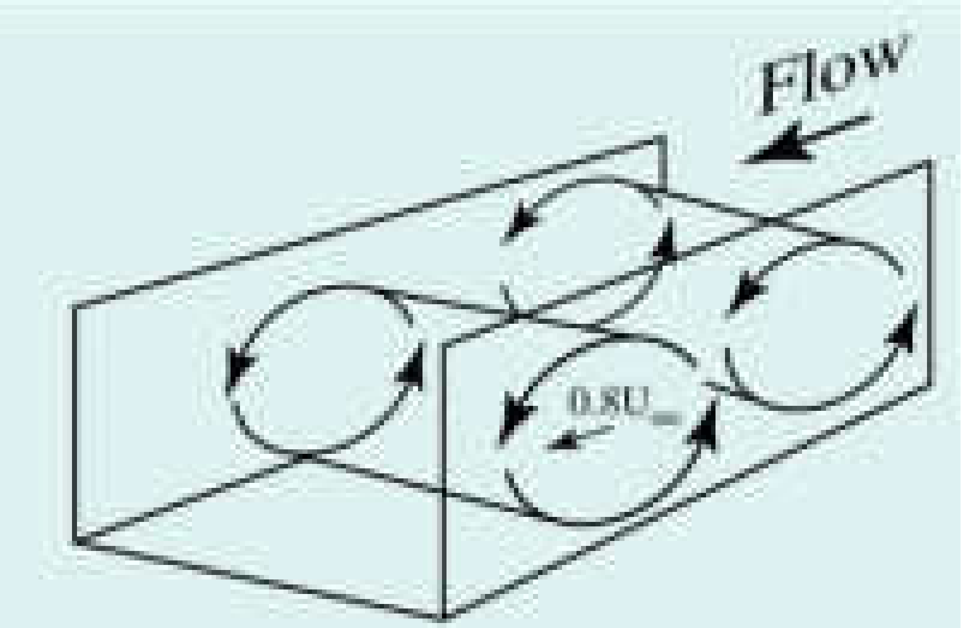


ii) Eddies

Components of turbulence that rotate about axes that are perpendicular to the mean flow direction.

Smaller scale than secondary flows and move downstream with the current at a speed of approximately 80% of the water surface velocity (U_∞).

Eddies move up and down within the flow as the travel downstream and lead to variation in boundary shear stress over time and along the flow direction.

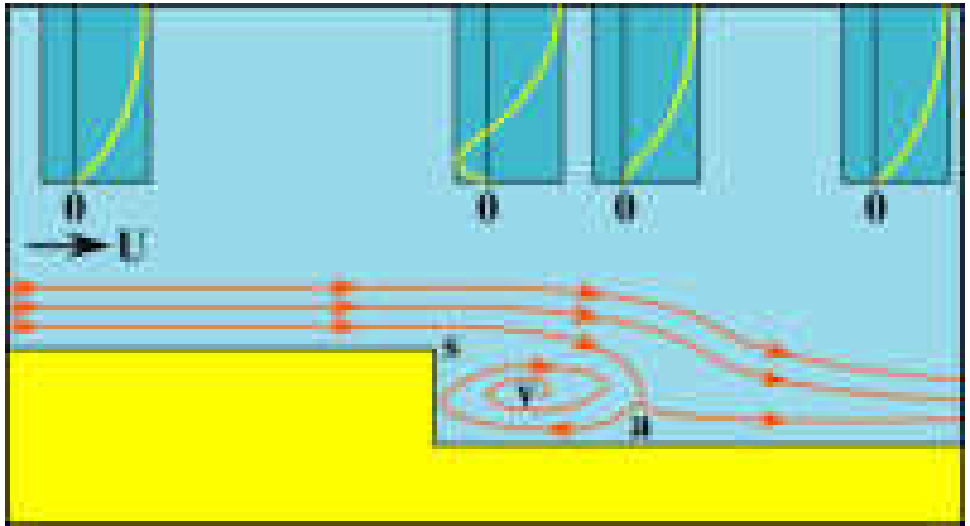


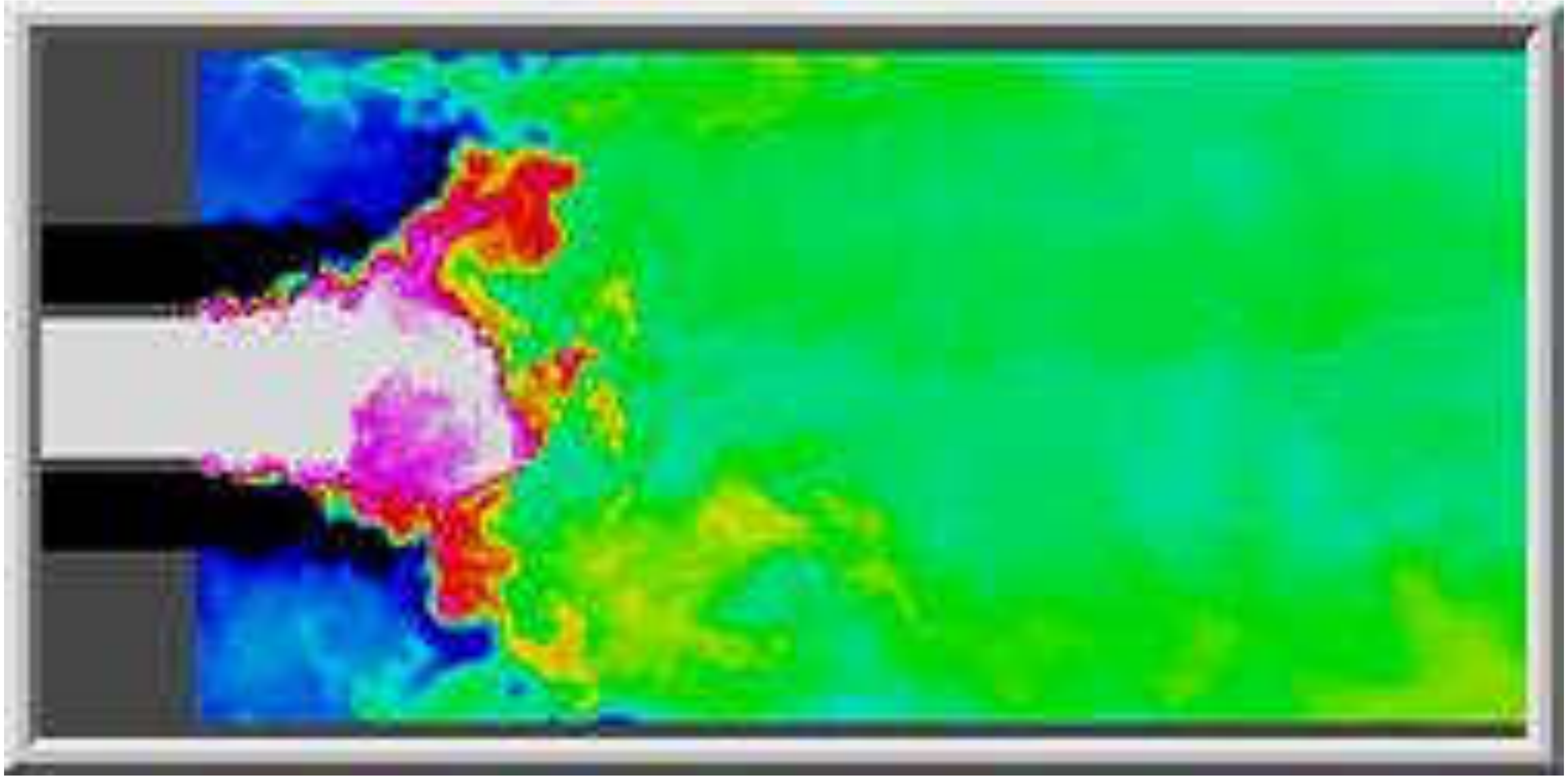
Some eddies are created by the topography of the bed.

In the lee of a negative step on the bed (see figure below) the flow separates from the boundary ("s" in the figure) and reattaches downstream ("a" in the figure).

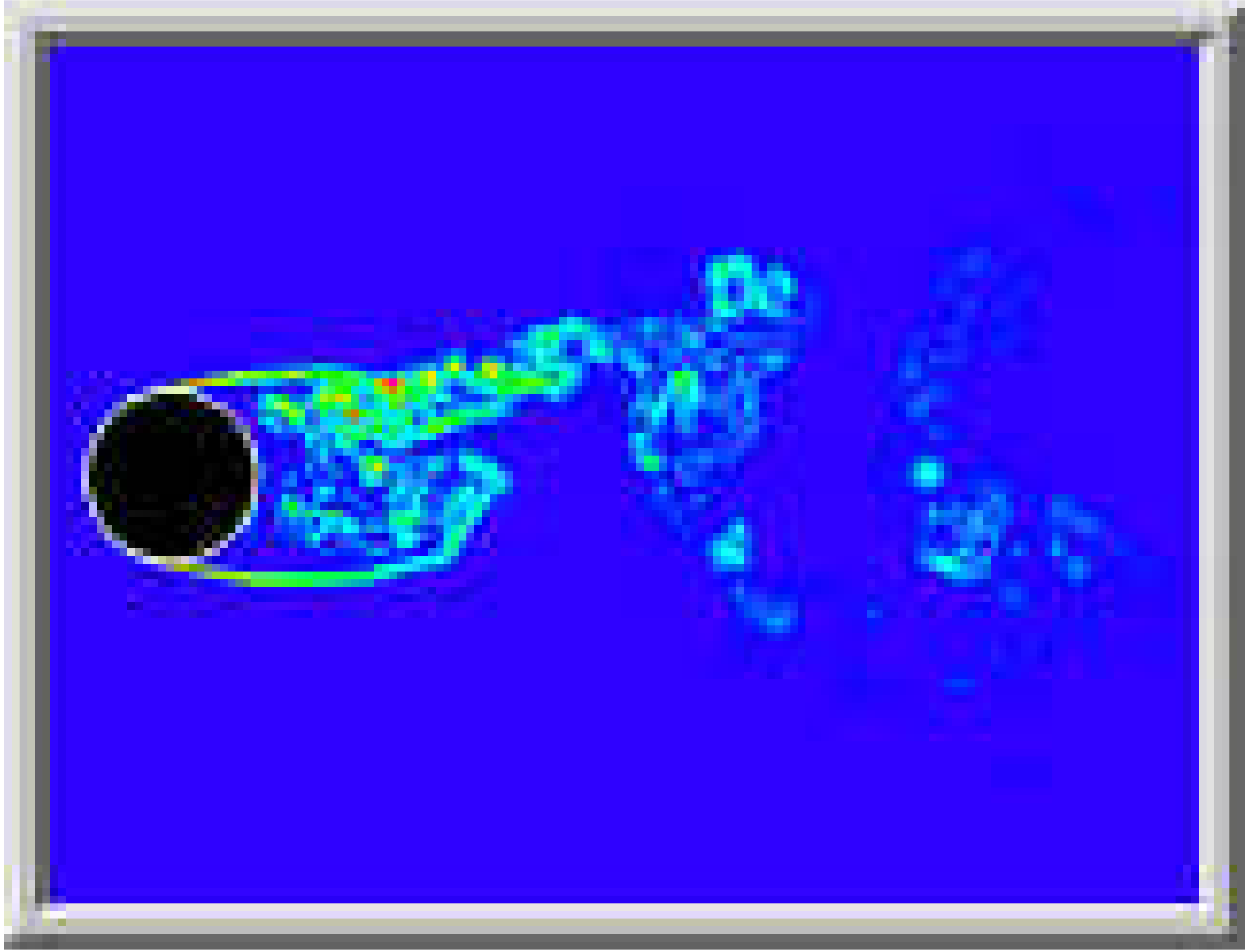
A *roller eddy* develops between the point of separation and the point of attachment.

Asymmetric bed forms (see next chapter) develop similar eddies.





[Flow over a step on the boundary.]



b) Small scale structures of the viscous sublayer.

i) Streaks

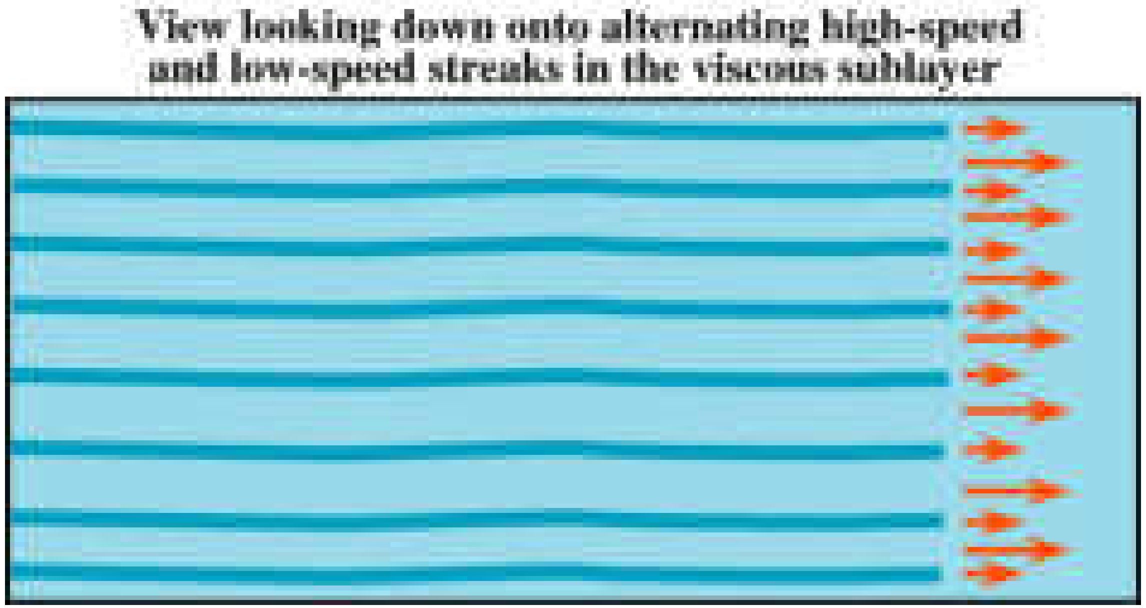
Alternating lanes of high and low speed fluid within the VSL.

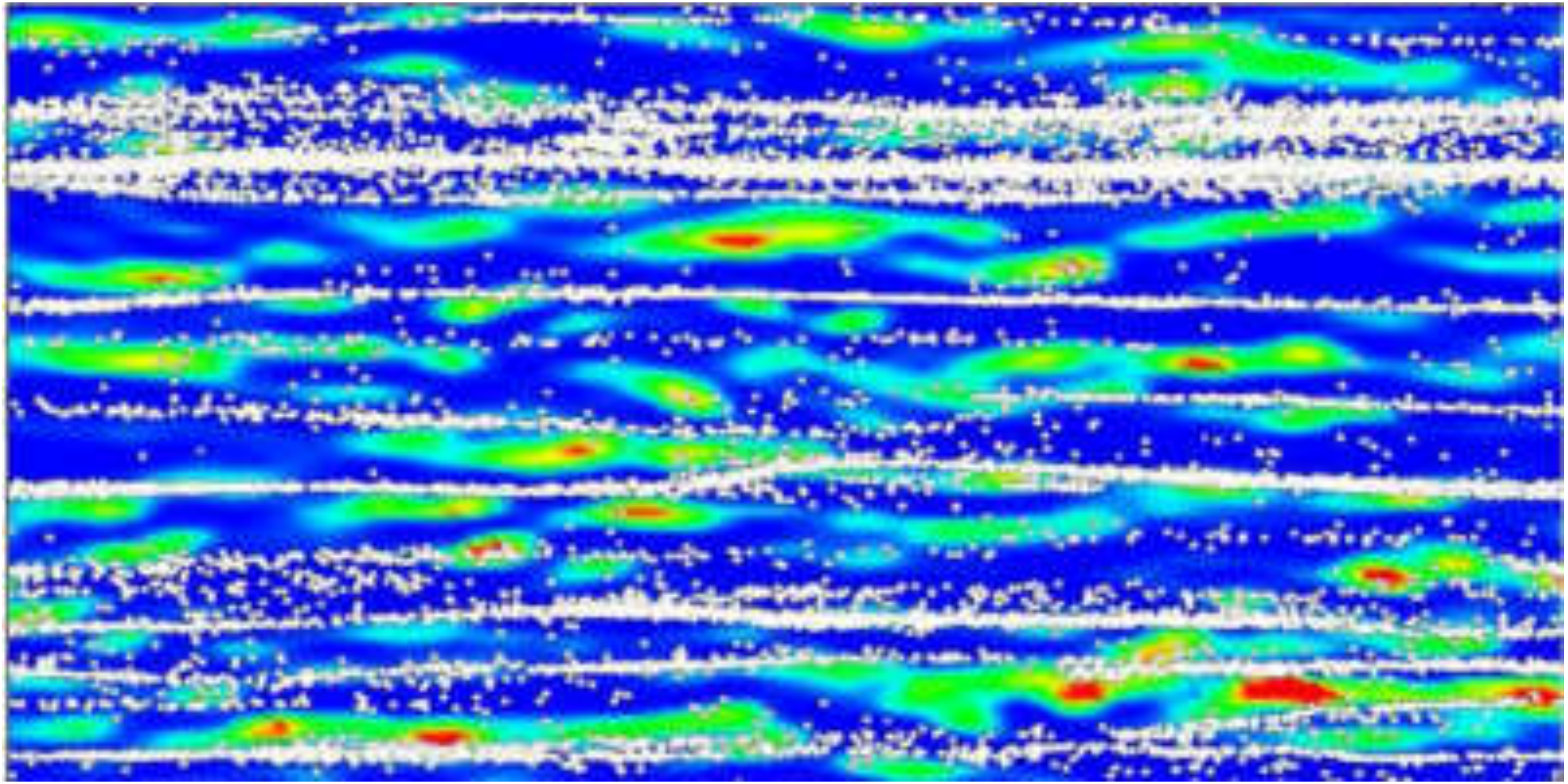
Associated with counter-rotating, flow parallel vortices within the VSL.

Streak spacing (λ) varies with the shear velocity and the kinematic viscosity of the fluid; λ ranges from millimetres to centimetres.

$$\lambda \approx \frac{100\nu}{U_*}$$

λ increases when sediment is present.





Red = high velocity
Blue = low velocity

ii) Bursts and sweeps

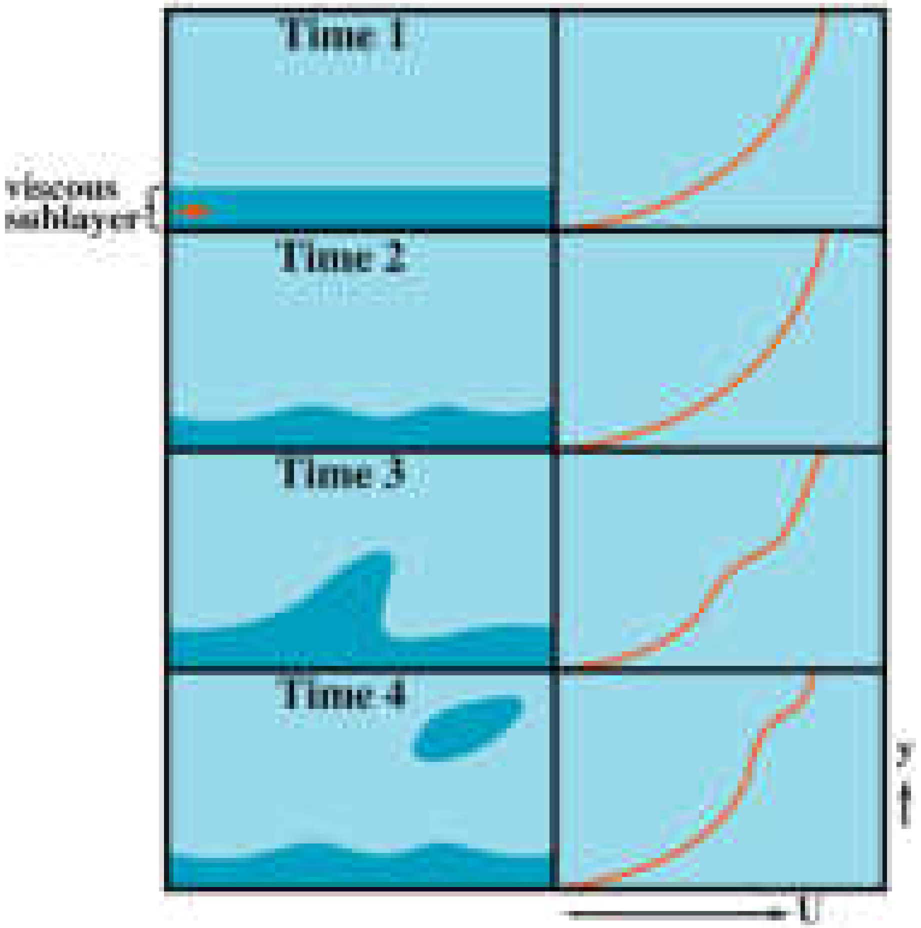
Burst: ejection of low speed fluid from the VSL into the outer layer.

Sweep: injection of **high** speed fluid from the outer layer into the VSL.

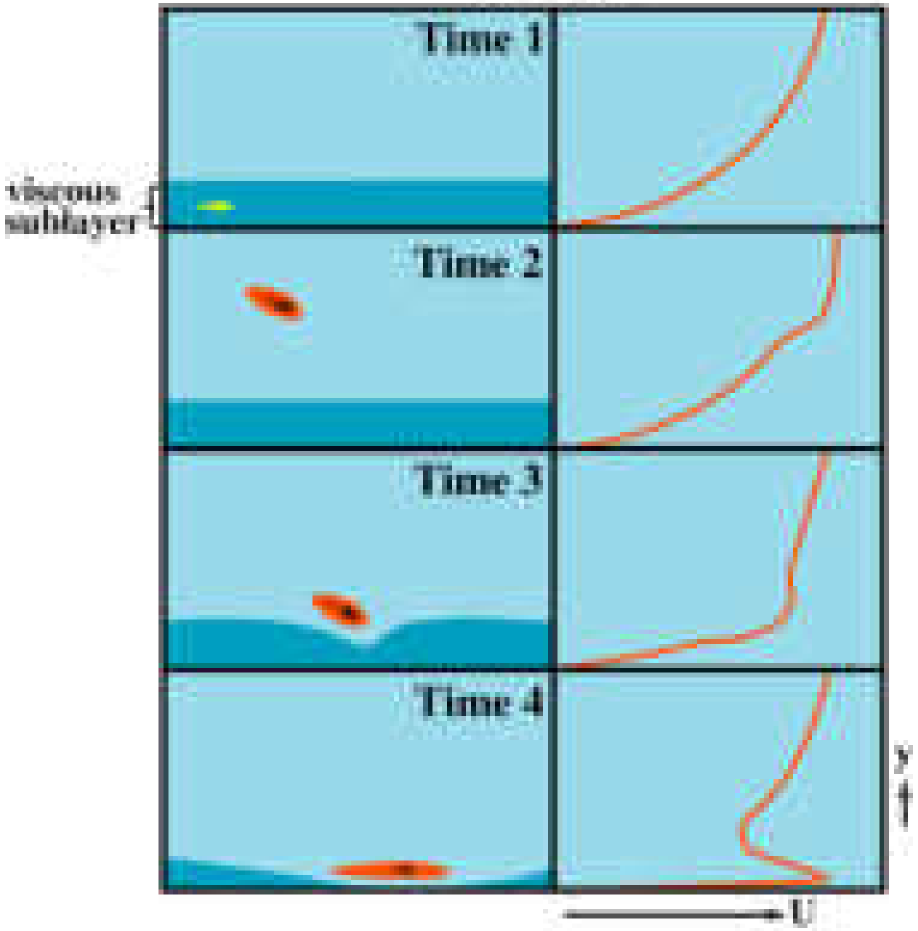
Often referred to as the “bursting cycle” but not every sweep causes a burst and vise versa.

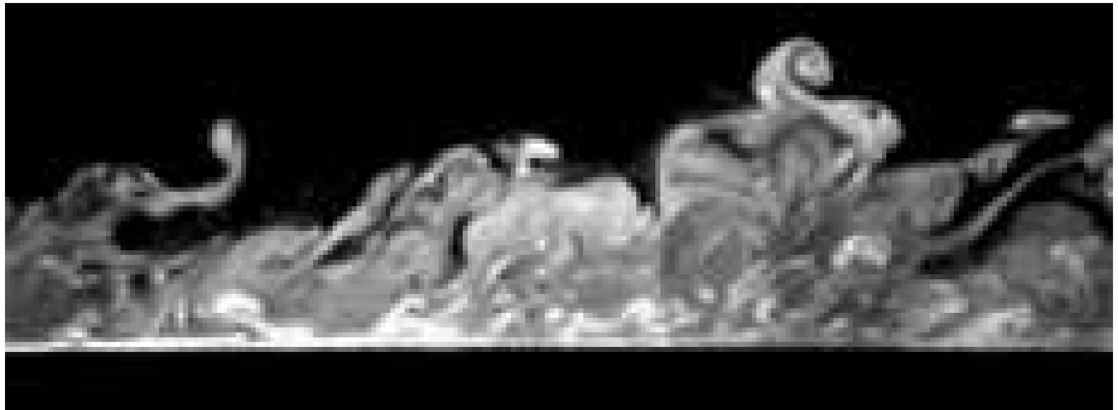
However, the frequency of bursting and sweeps are approximately equal.

A burst



A Sweep





I. Classification of sediment load

The sediment that is transported by a current.

Two main classes:

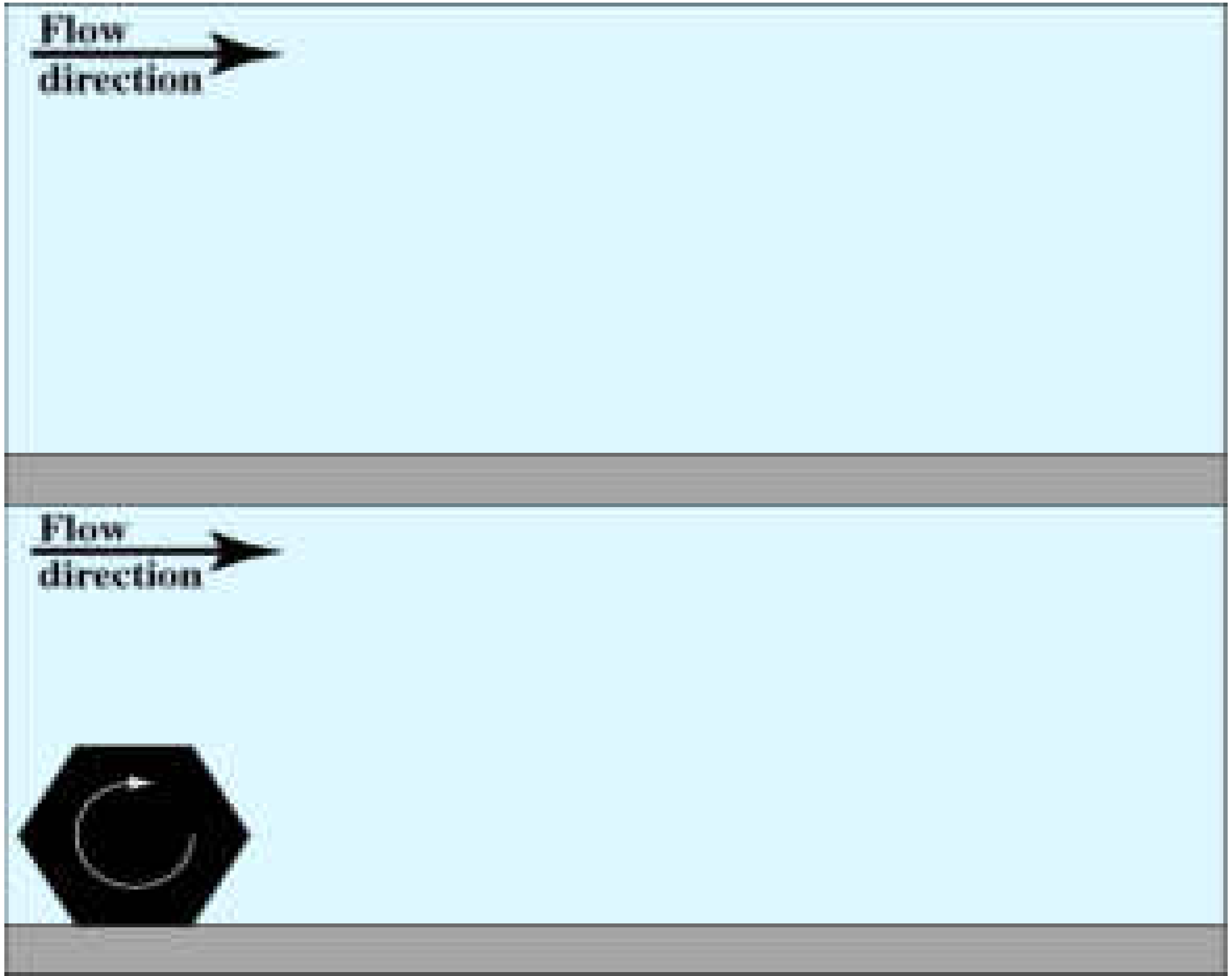
Wash load: silt and clay size material that remains in suspension even during low flow events in a river.

Bed material load: sediment (sand and gravel size) that resides in the bed but goes into transport during high flow events (e.g., floods).

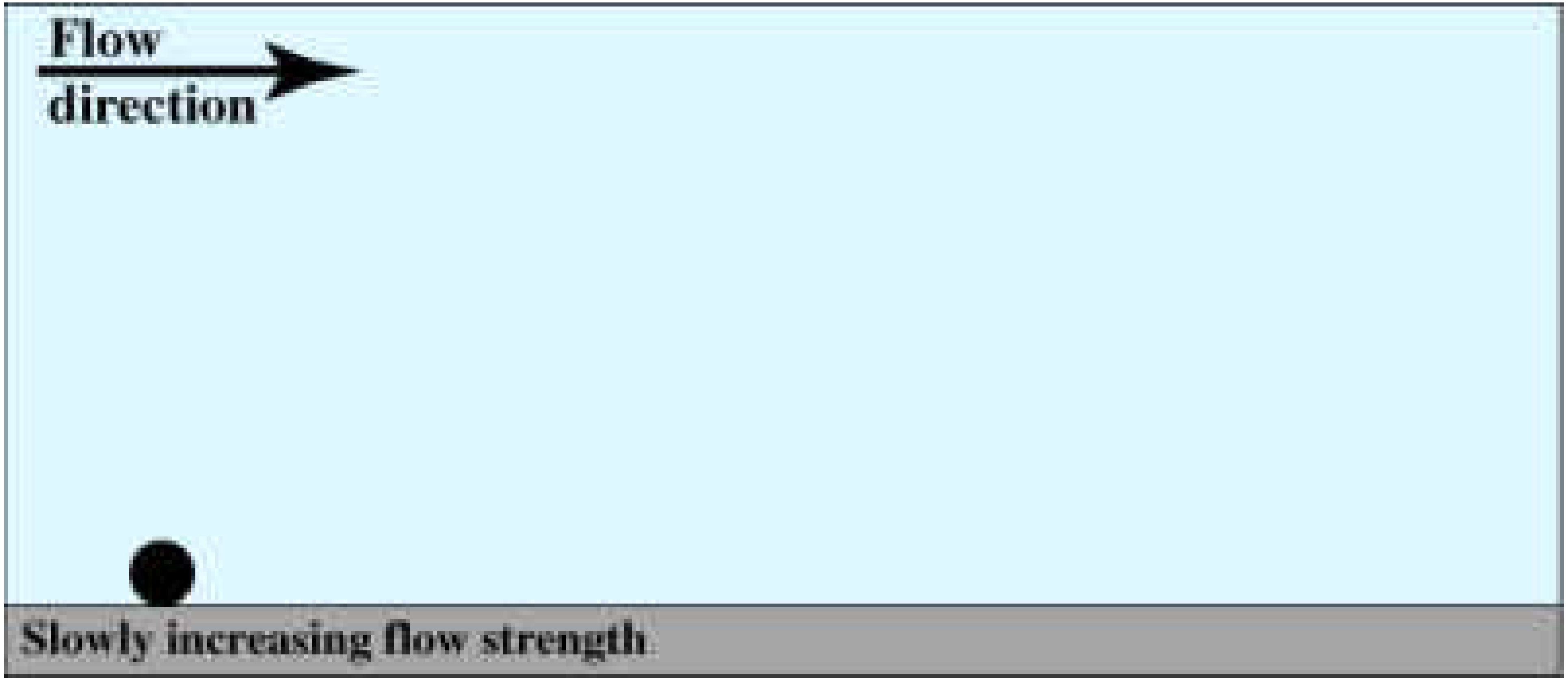
Bed material load makes up many arenites and rudites in the geological record.

Three main components of bed material load.

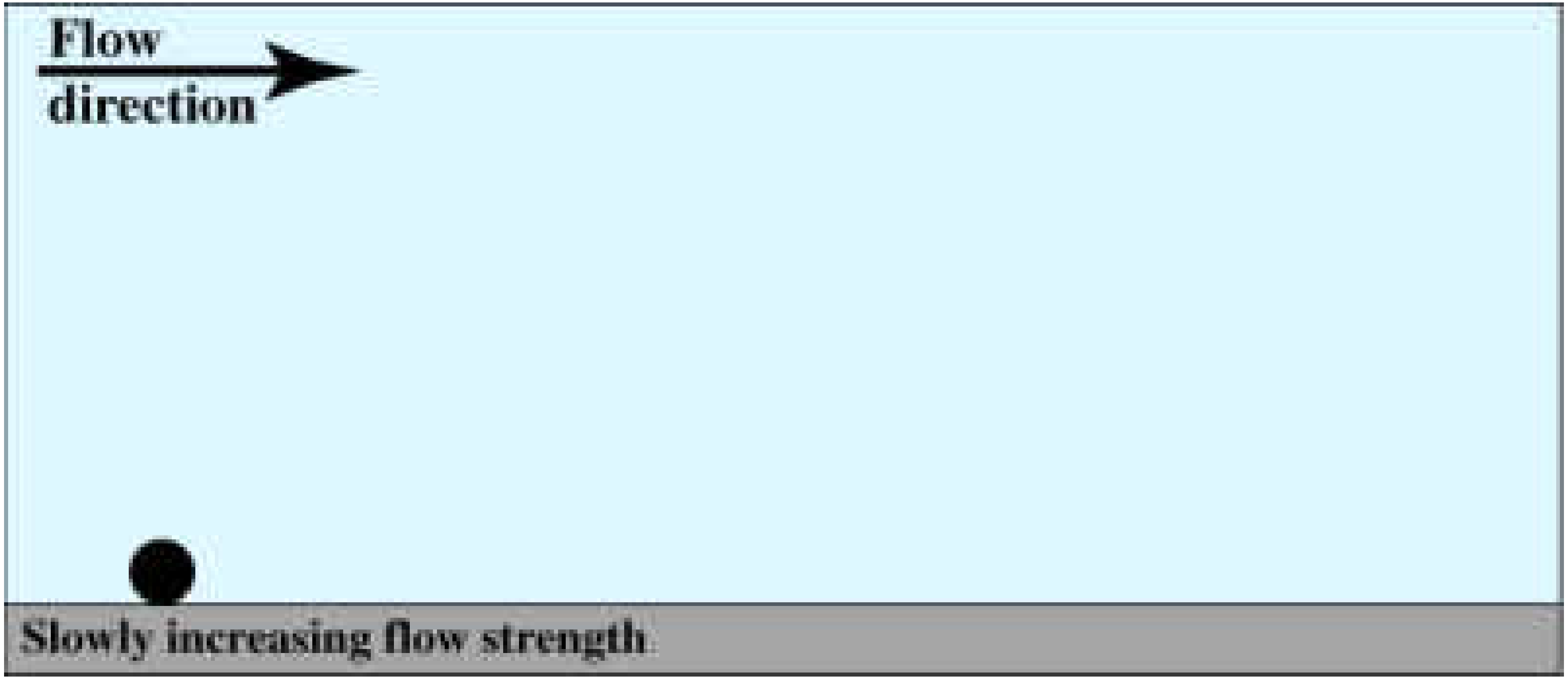
Contact load: particles that move in contact with the bed by sliding or rolling over it.



Saltation load: movement as a series of "hops" along the bed, each hop following a ballistic trajectory.



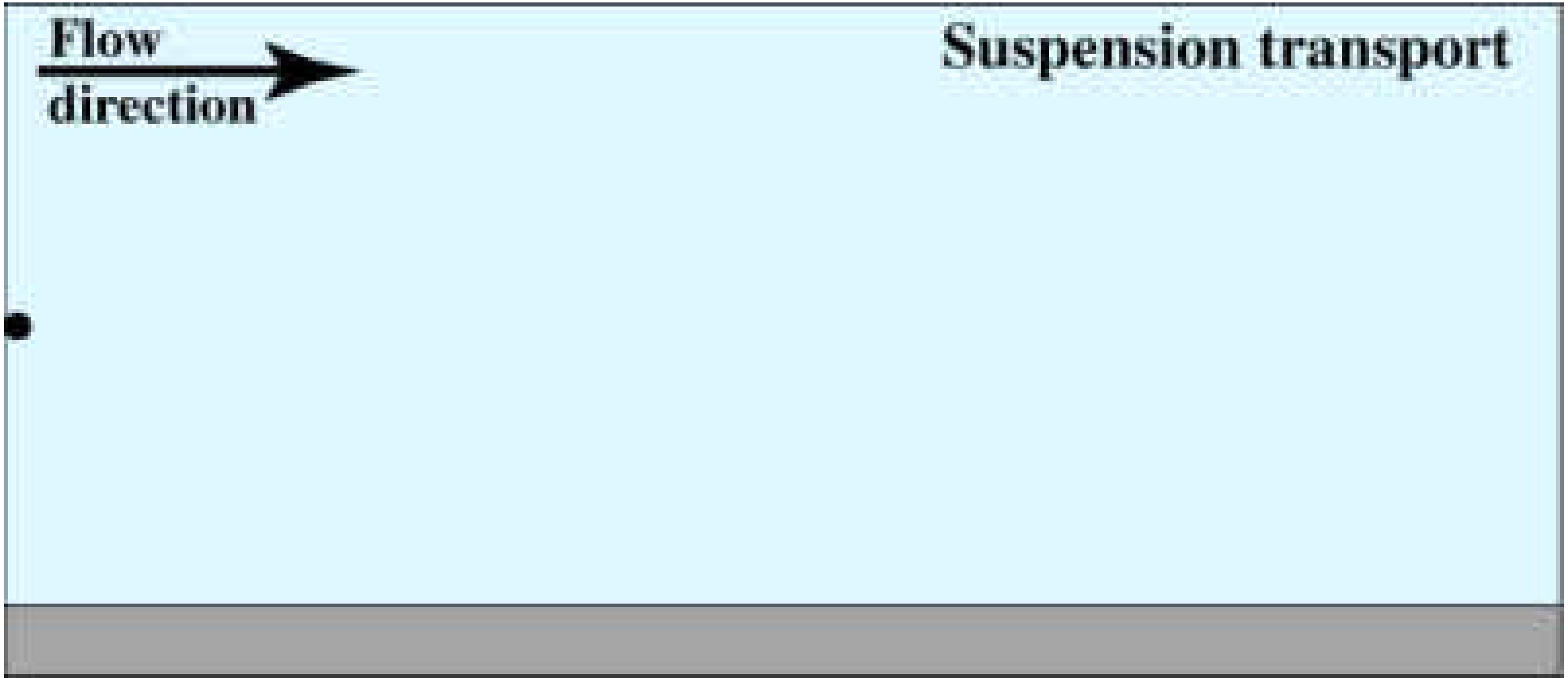
When the ballistic trajectory is disturbed by turbulence the motion is referred to as *Suspensive saltation.*



Intermittent suspension load: carried in suspension by turbulence in the flow.

“Intermittent” because it is in suspension only during high flow events and otherwise resides in the deposits of the bed.

Bursting is an important process in initiating suspension transport.

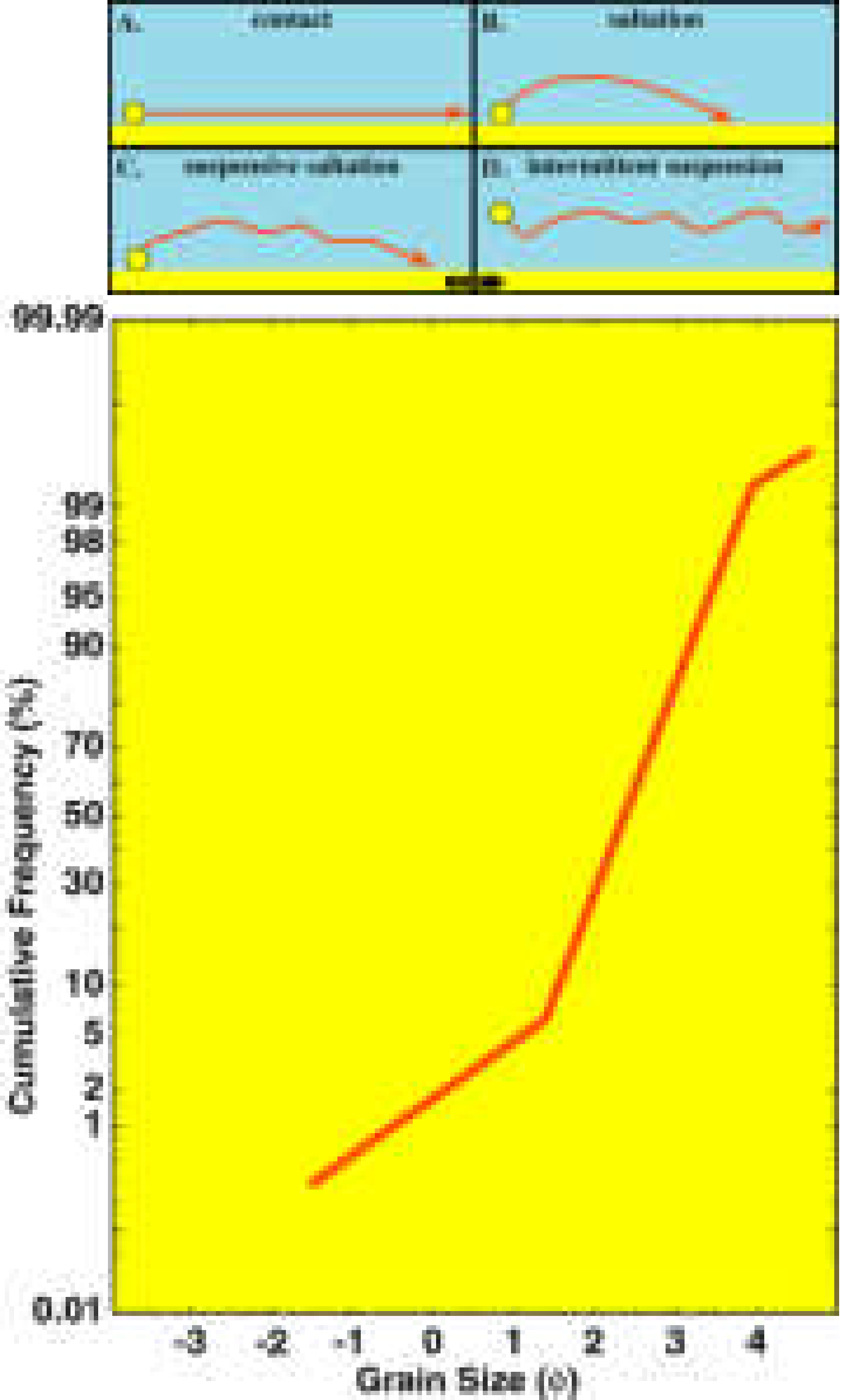


A very nice java applet showing movement on the bed in response to an incoming sweep.

II. Hydraulic interpretation of grain size distributions

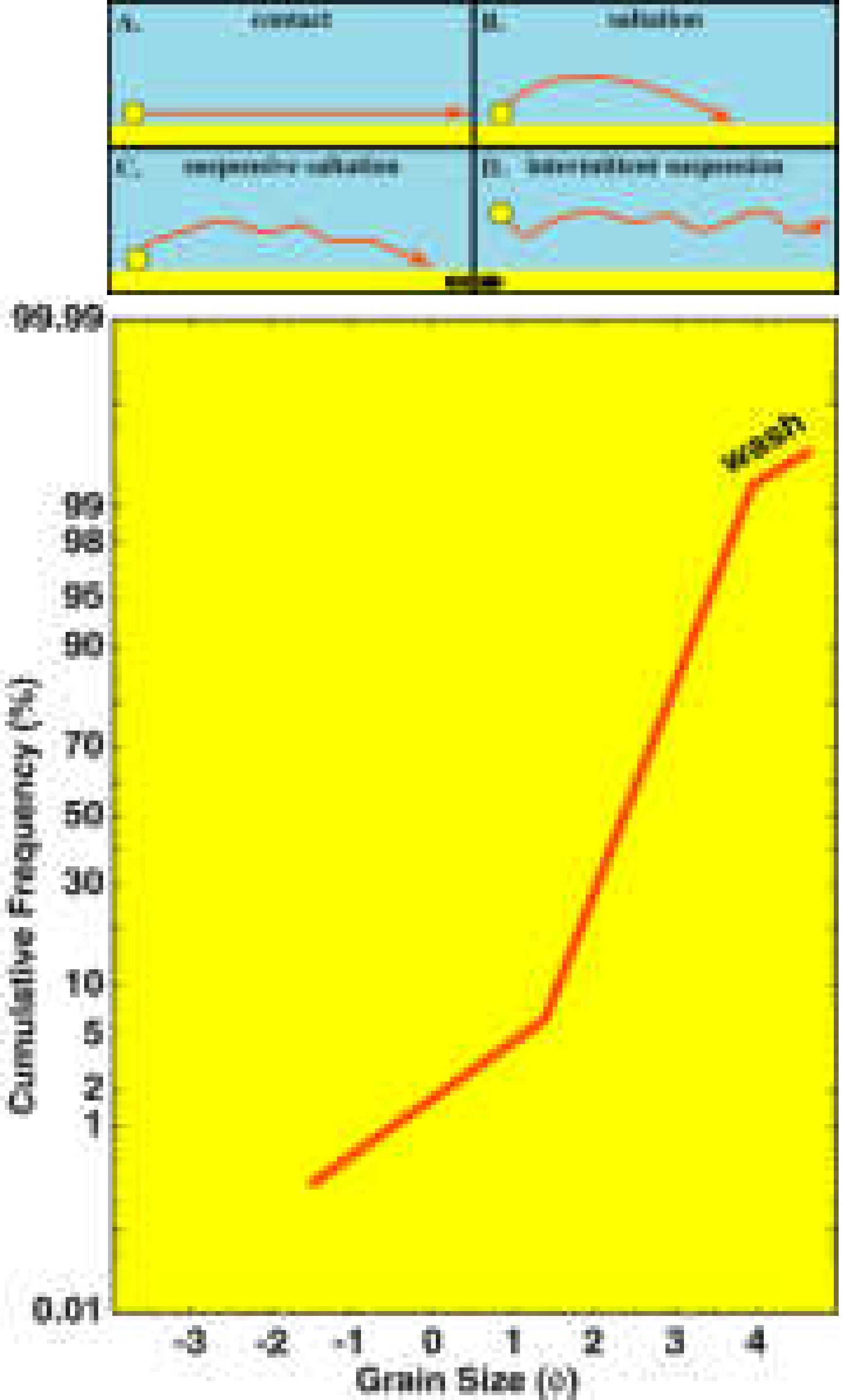
In the section on grain size distributions we saw that some sands are made up of several normally distributed subpopulations.

These subpopulations can be interpreted in terms of the modes of transport that they underwent prior to deposition.



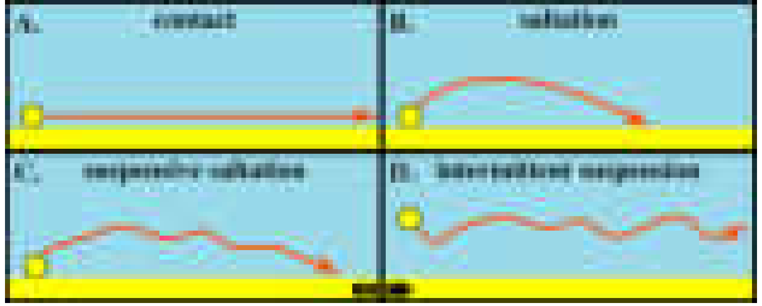
The finest subpopulation represents the wash load.

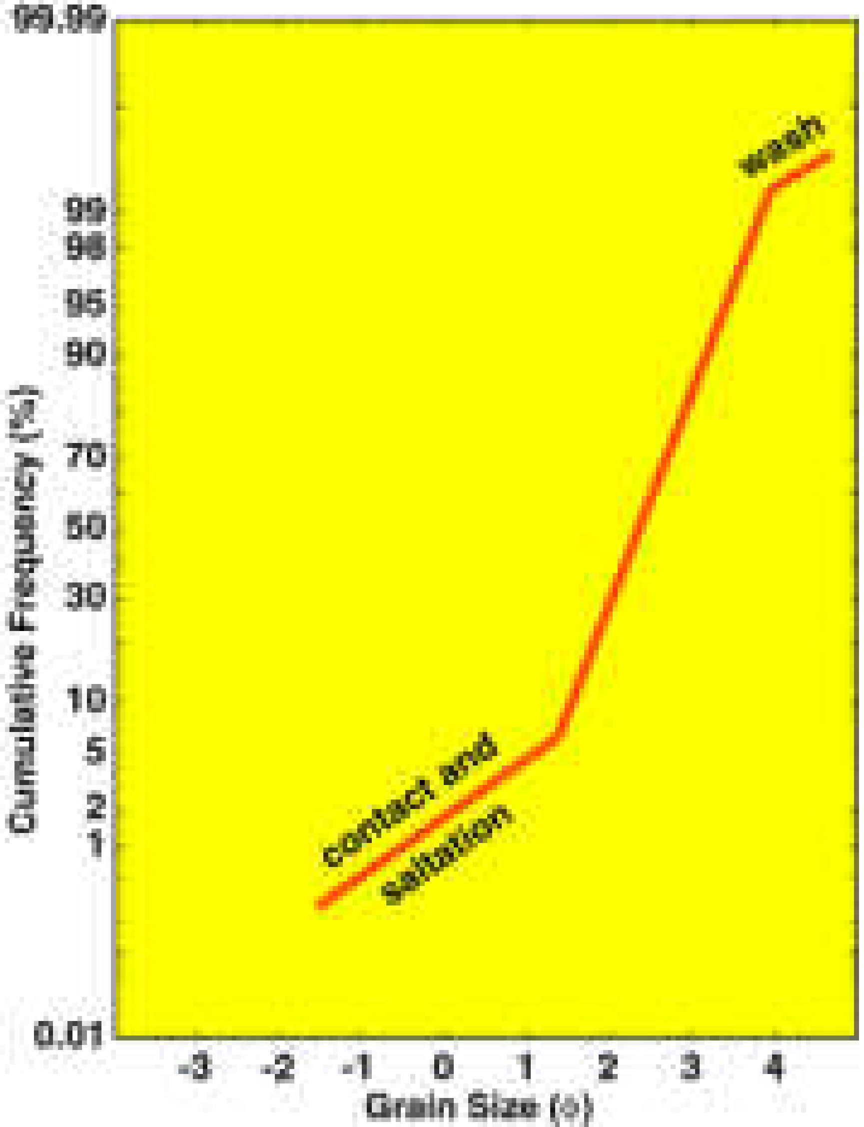
Only a very small amount of wash load is ever stored within the bed material so that it makes up a very small proportion of these deposits.



The coarsest subpopulation represents the contact and saltation loads.

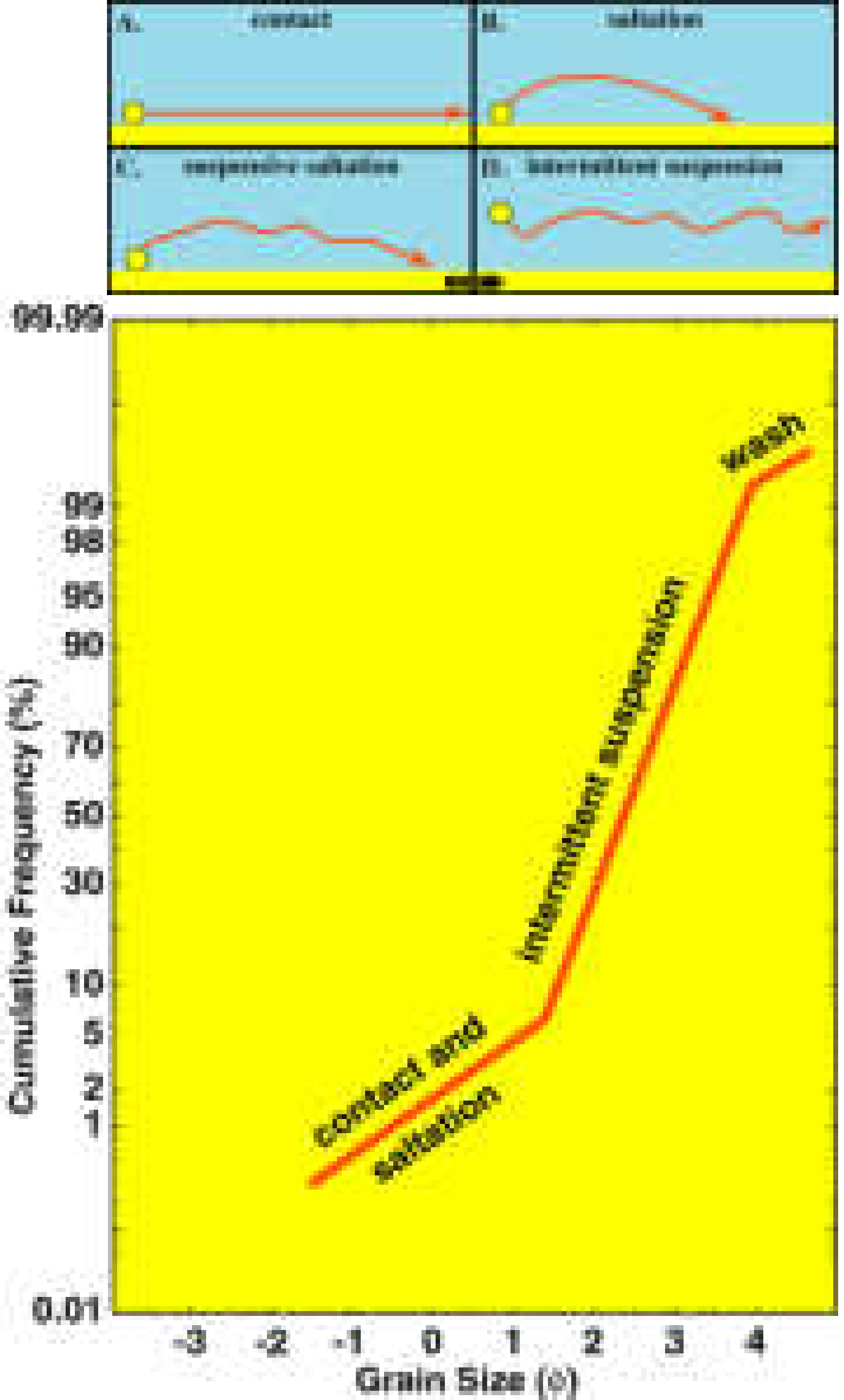
In some cases they make up two subpopulations (only one is shown in the figure).





The remainder of the distribution, normally making up the largest proportion, is the intermittent suspension load.

This interpretation of the subpopulations gives us two bases for quantitatively determining the strength of the currents that transported the deposits.



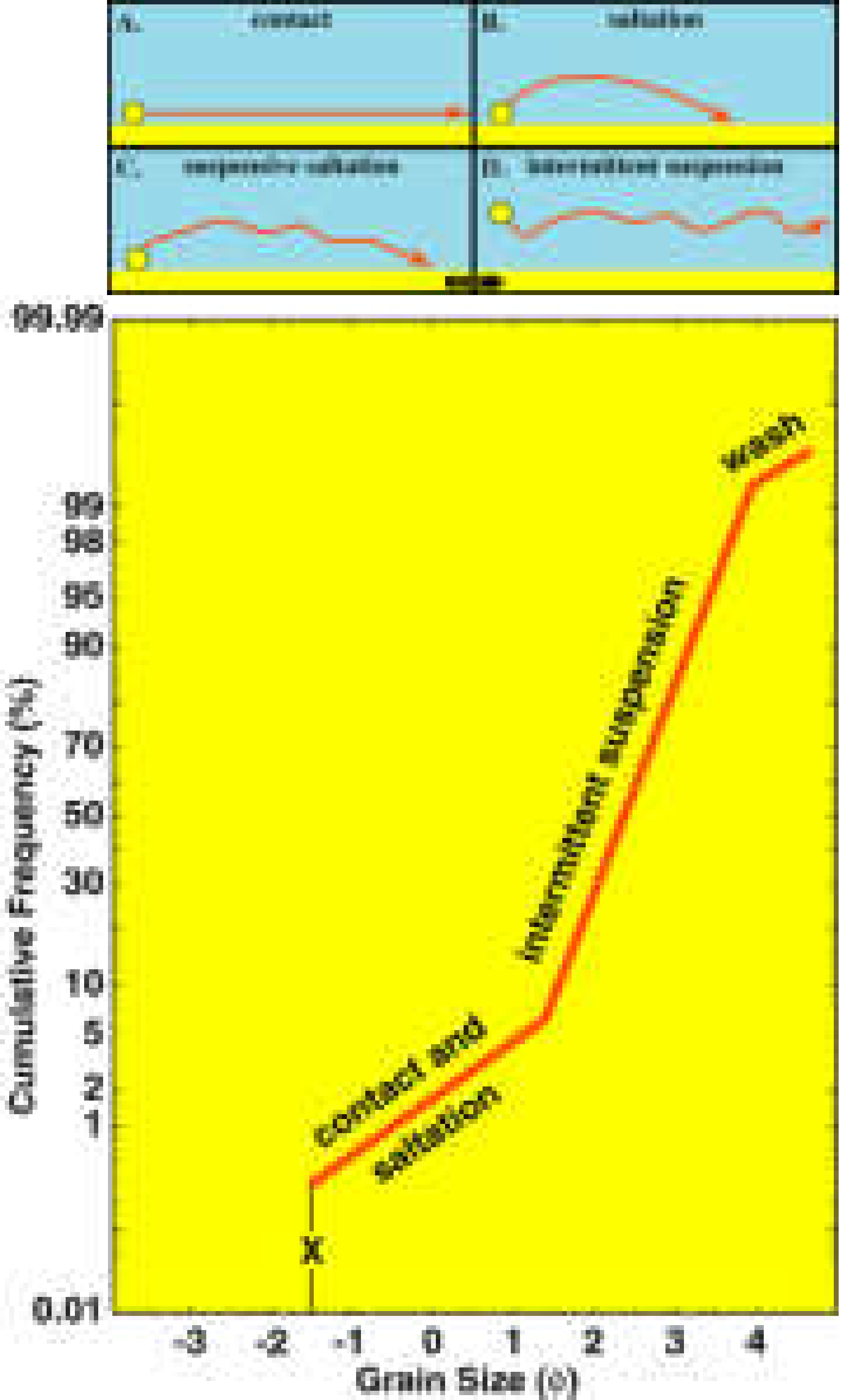
The grain size “X” is the coarsest sediment that the currents could move on the bed.

In this case, X = -1.5 ϕ or approximately 2.8 mm.

If the currents were weaker, that grain size would not be present.

If the currents were stronger, coarser material would be present.

This assumes that there were no limitations to the size of grains available in the system.



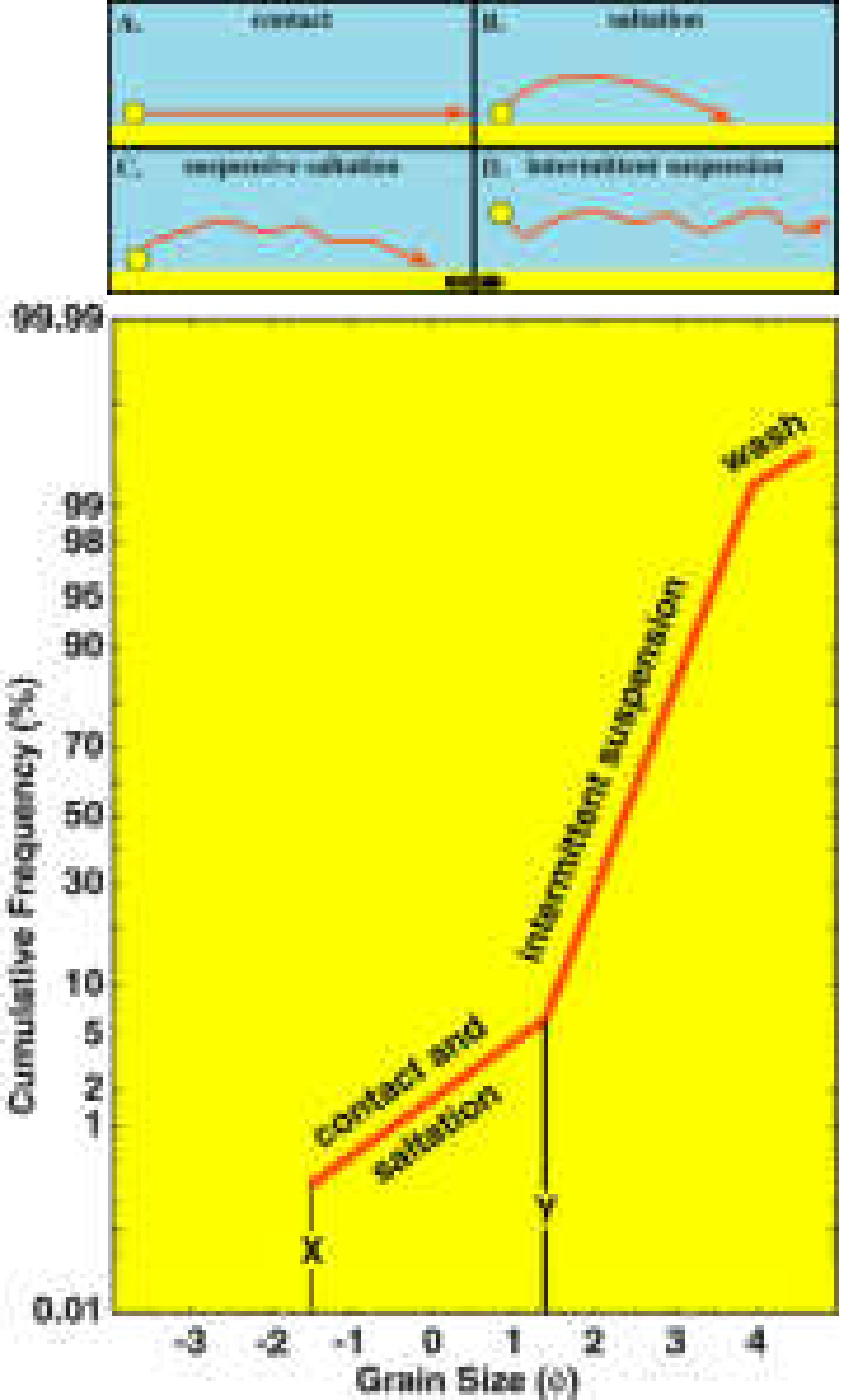
The grain size “Y” is the coarsest sediment that the currents could take into suspension.

In this case, Y = 1.3 ϕ or approximately 0.41 mm.

Therefore the currents must have been just powerful enough to take the 0.41 mm particles into suspension.

If the currents were stronger the coarsest grain size would be larger.

This assumes that there were no limitations to the size of grains available in the system.

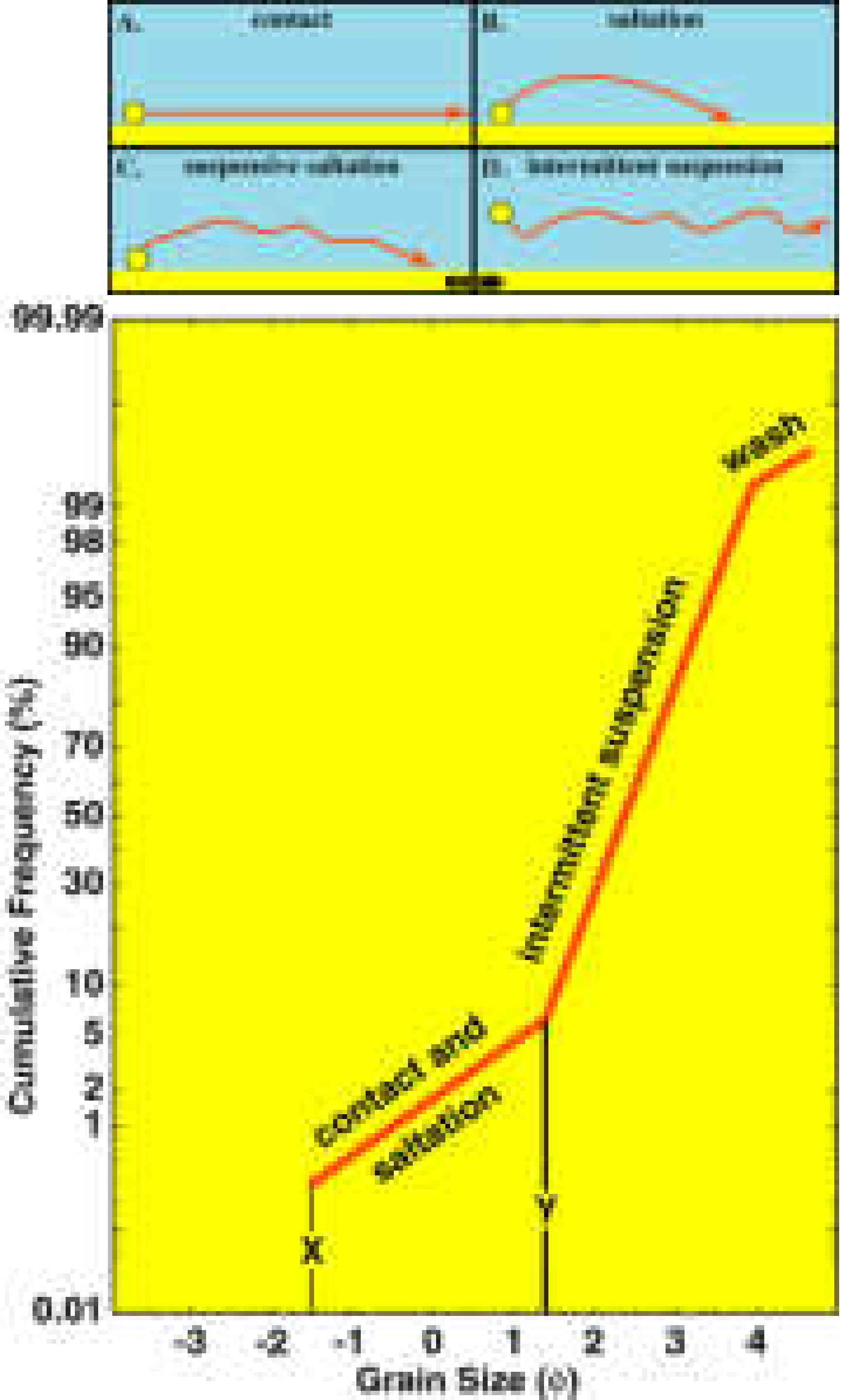


To quantitatively interpret “X” we need to know the hydraulic conditions needed to just begin to move of that size.

This condition is the “threshold for sediment movement”.

To quantitatively interpret “Y” we need to know the hydraulic conditions needed to just begin carry that grain size in suspension.

This condition is the “threshold for suspension”.



a) The threshold for grain movement on the bed.

Grain size “X” can be interpreted if we know what flow strength is required to just move a particle of that size.

That flow strength will have transported sediment with that maximum grain size.

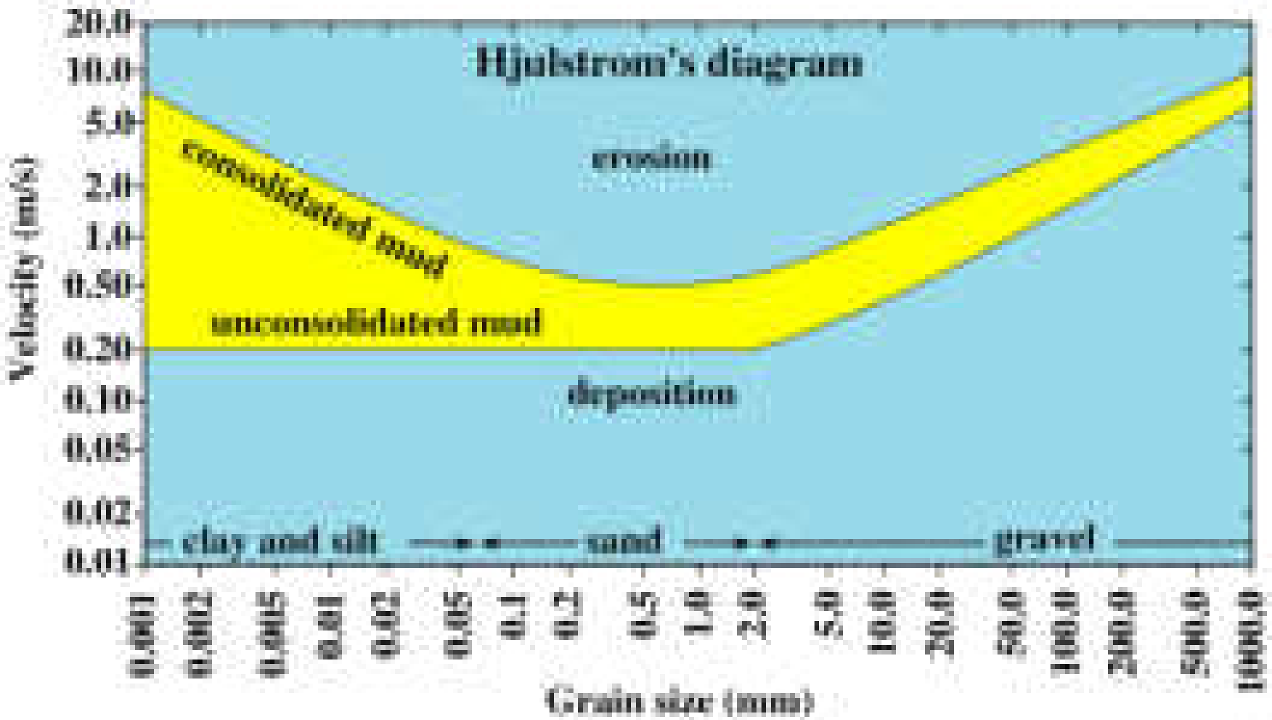
Several approaches have been taken to determine the critical flow strength to initiate motion on the bed.

i) Hjulstrom's Diagram

Based on a series of experiments using unidirectional currents with a flow depth of 1 m.

The diagram (below) shows the critical velocity that is required to just begin to move sediment of a given size (the top of the yellow field).

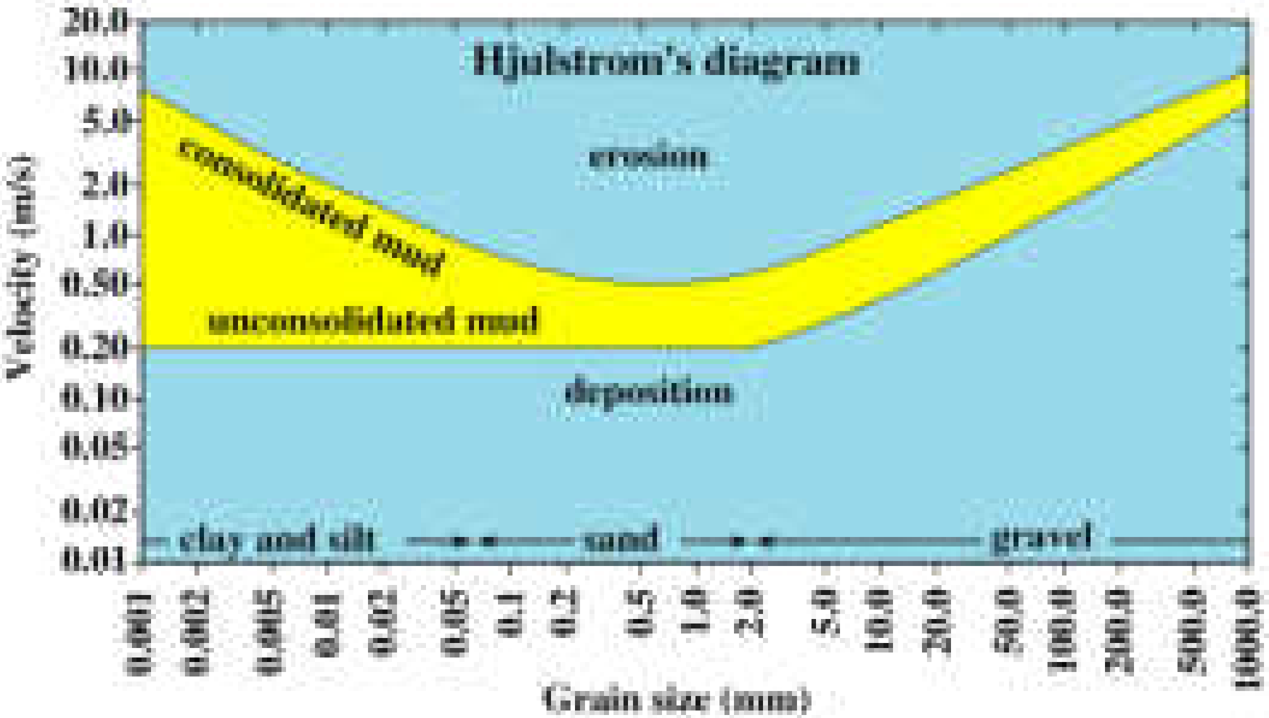
It also shows the critical velocity for deposition of sediment of a given size (the bottom of the yellow field).



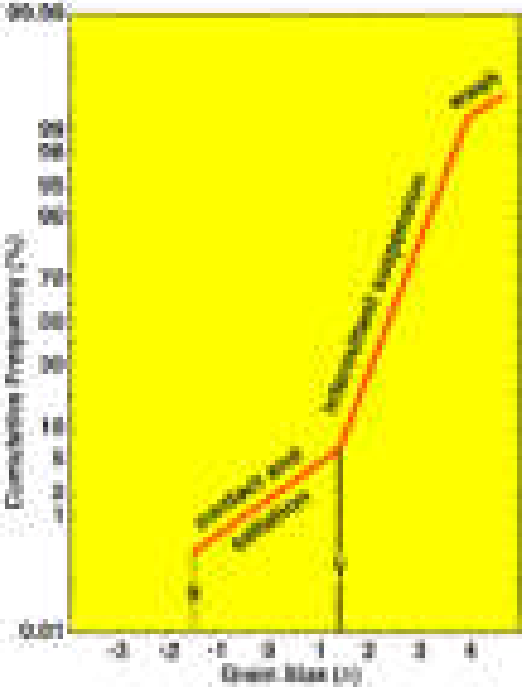
Note that for grain sizes coarser than 0.5 mm the velocity that is required for transport increases with grain size; the larger the particles the higher velocity that is required for transport.

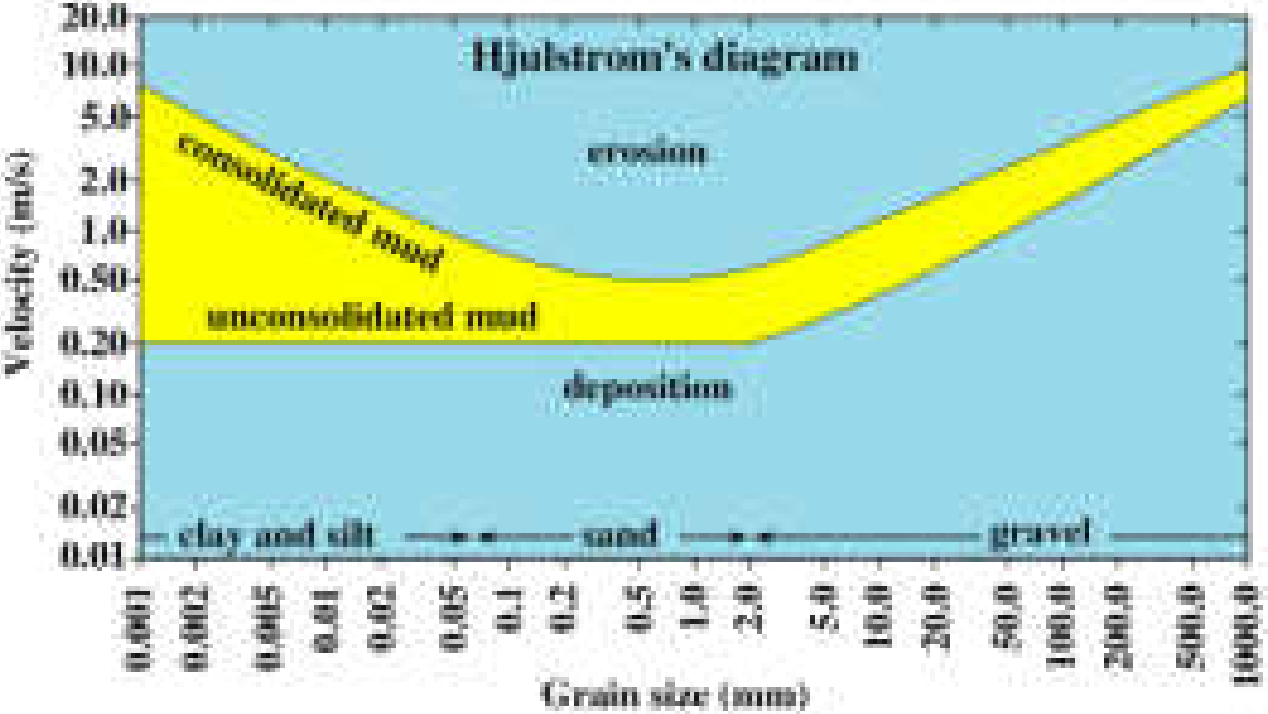
For finer grain sizes (with cohesive clay minerals) the finer the grain size the greater the critical velocity for transport.

This is because the more mud is present the greater the cohesion and the greater the resistance to erosion, despite the finer grain size.



In our example, the coarsest grain size was 2.8 mm.

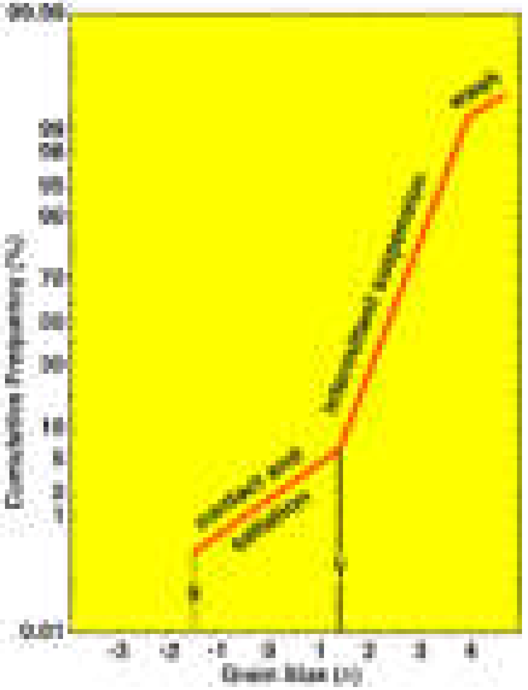


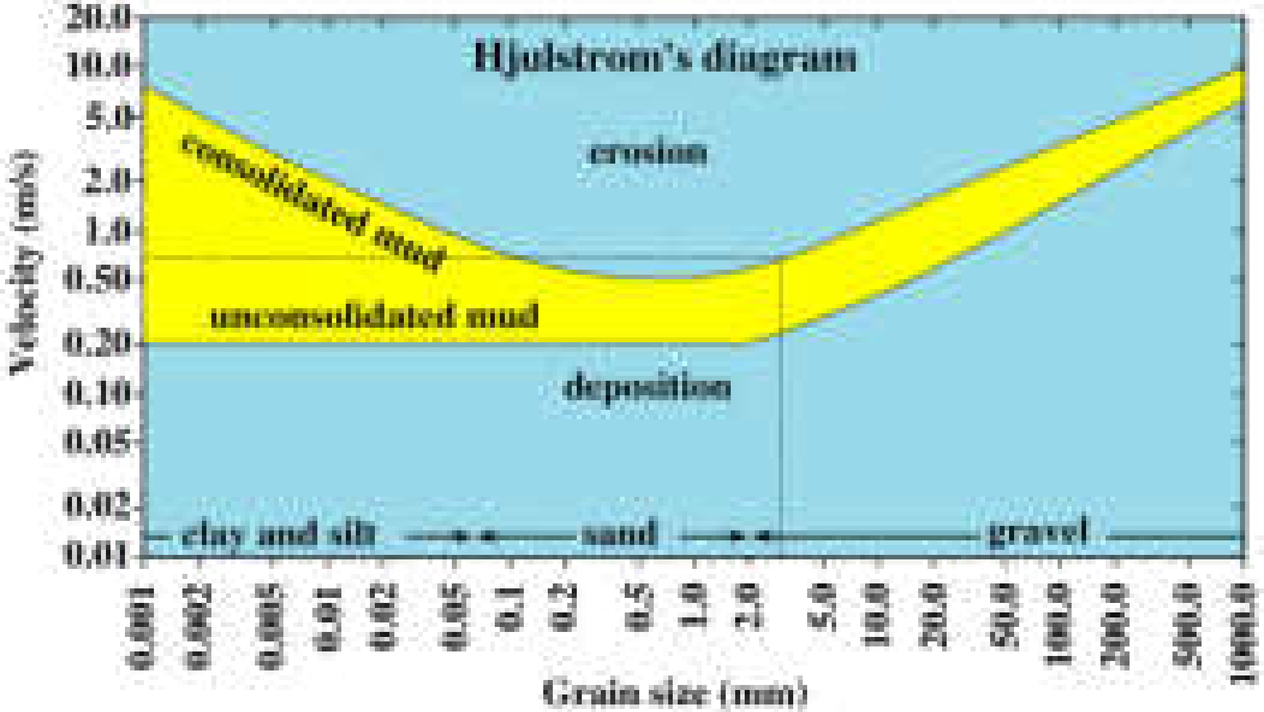


In our example, the coarsest grain size was 2.8 mm.

According to Hjulstron's diagram, that grain size would require a flow with a velocity of approximately 0.65m/s.

Therefore, the sediment shown in the cumulative frequency curve was transported by currents at 0.65 m/s.





The problem is that the forces that are required to move sediment are not only related to flow velocity.

Boundary shear stress is a particularly important force and it varies with flow depth.

$$\tau_o = \rho g D \sin\theta$$

Therefore, Hjulstrom's diagram is reasonably accurate only for sediment that has been deposited under flow depths of 1 m.

i) Shield's criterion for the initiation of motion

Based on a large number of experiments Shield's criterion considers the problem in terms of the forces that act to move a particle.

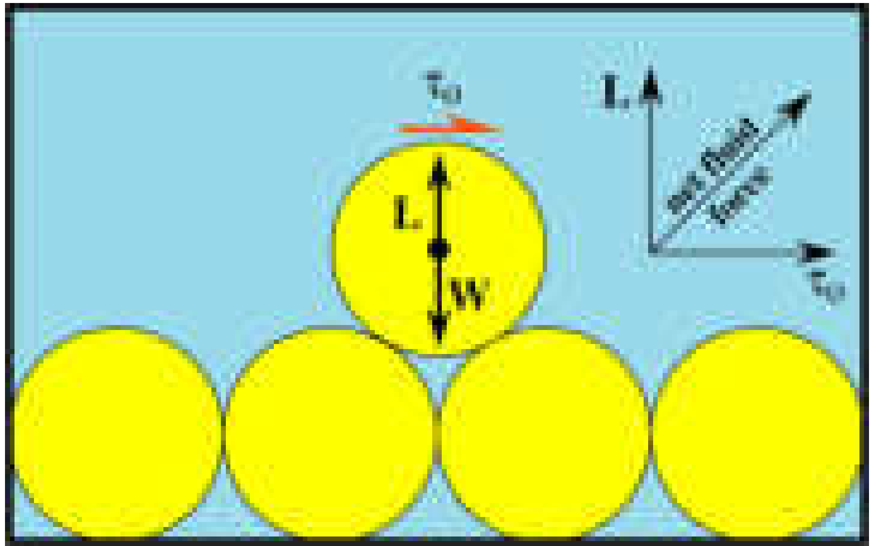
The criterion applies to beds of spherical particles of uniform grain size.

Forces that are important to initial motion:

1. The submerged weight of the particle ($\frac{\pi}{6}(\rho_s - \rho)gd^3$) which resists motion.

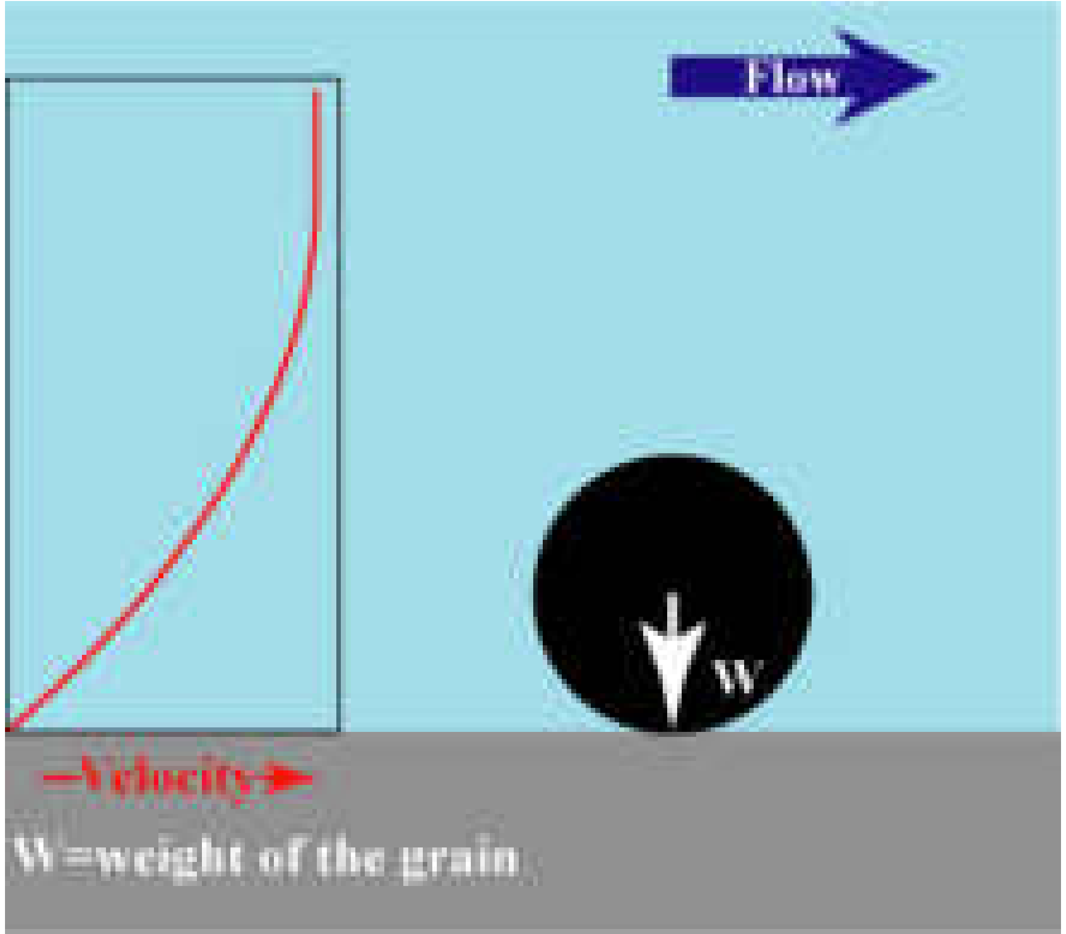
2. τ_o which causes a drag force that acts to move the particle down current.

3. Lift force (L) that reduces the effective submerged weight.



What’s a Lift Force?

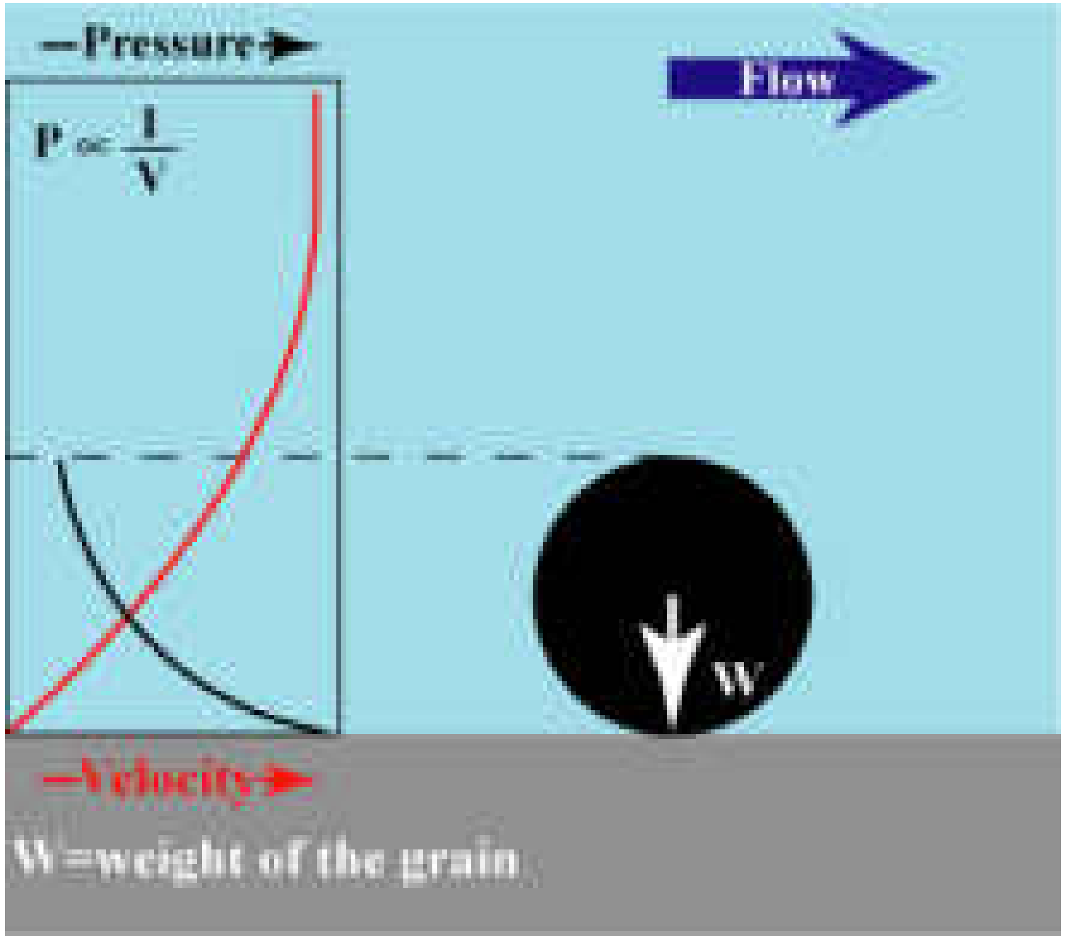
The flow velocity that is “felt” by the particle varies from approximately zero at its base to some higher velocity at its highest point.



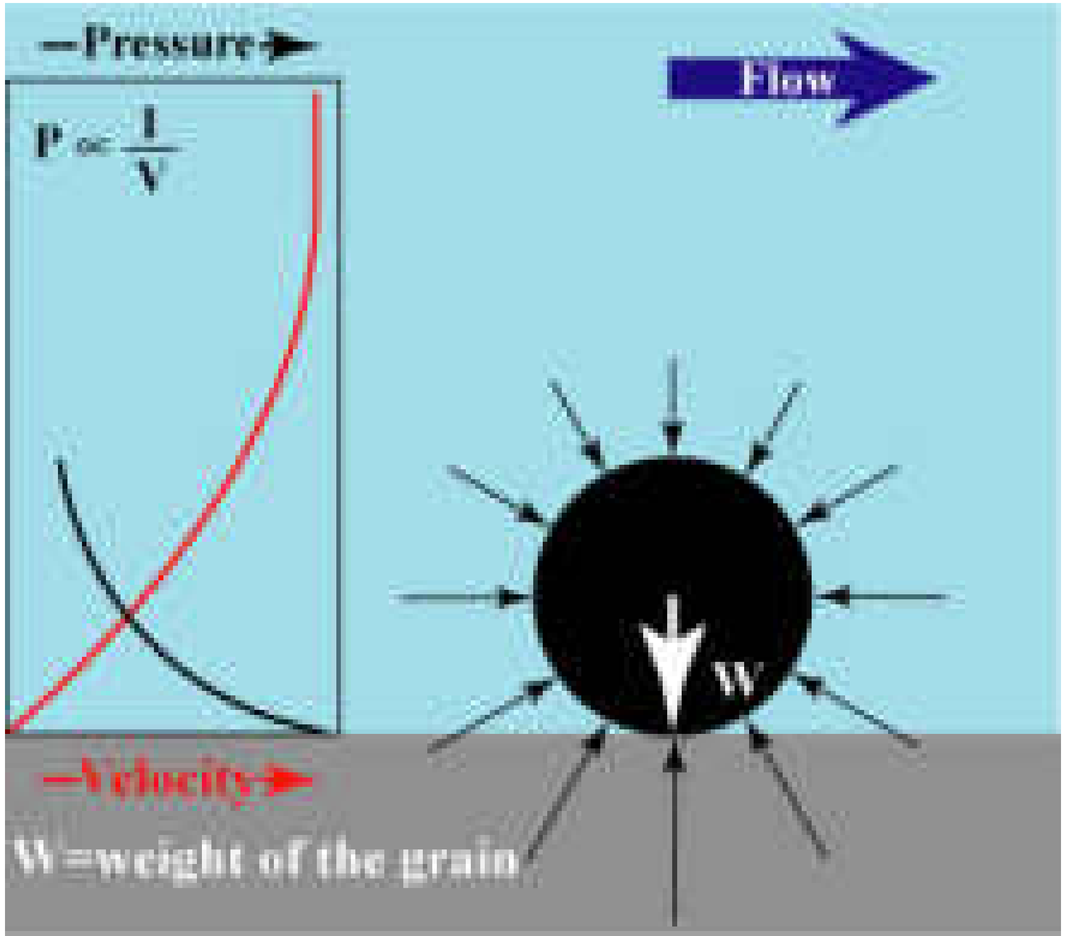
Pressure (specifically "dynamic pressure" in contrast to static pressure) is also imposed on the particle and the magnitude of the dynamic pressure varies inversely with the velocity:

Higher velocity, lower dynamic pressure.

Maximum dynamic pressure is exerted at the base of the particle and minimum pressure at its highest point.



The dynamic pressure on the particle varies symmetrically from a minimum at the top to a maximum at the base of the particle.

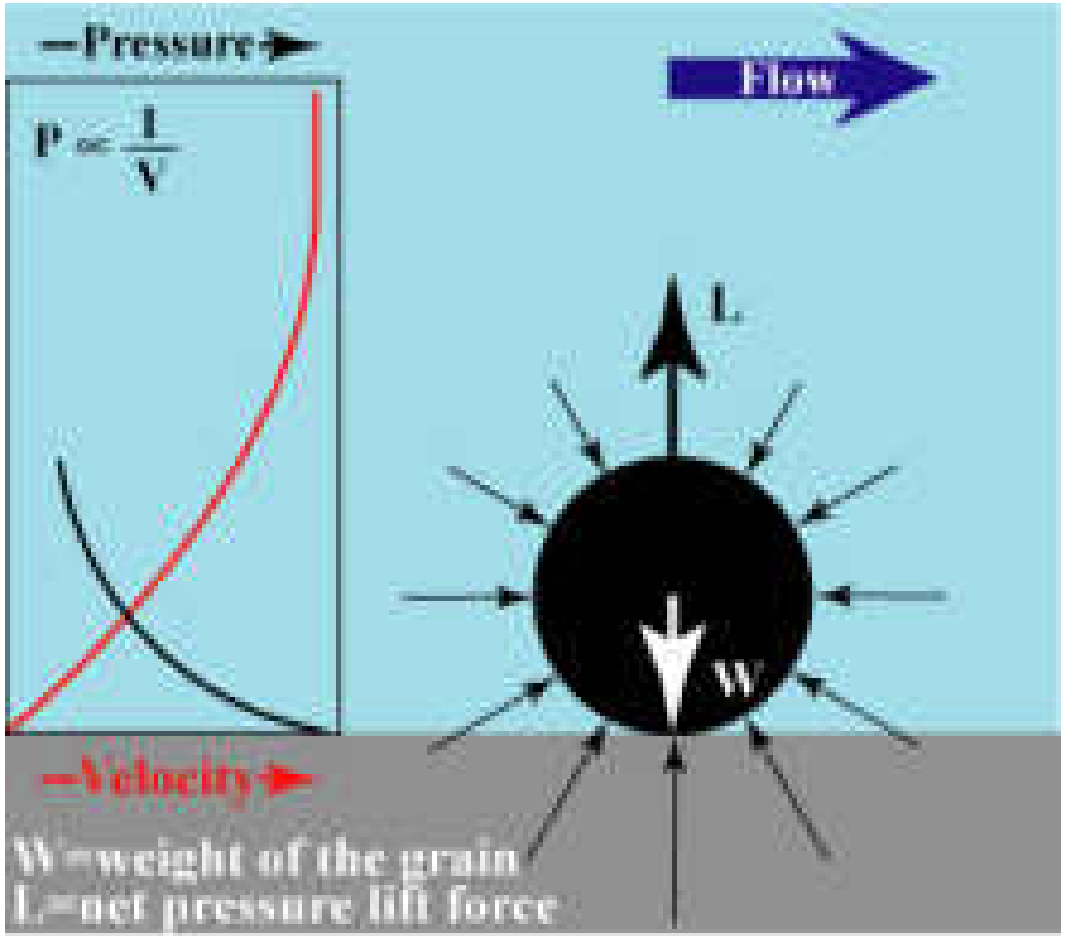


This distribution of dynamic pressure results in a net pressure force that acts upwards.

Thus, the net pressure force (known as the Lift Force) acts oppose the weight of the particle (reducing its effective weight).

This makes it easier for the flow to roll the particle along the bed.

The lift force reduces the drag force that is required to move the particle.

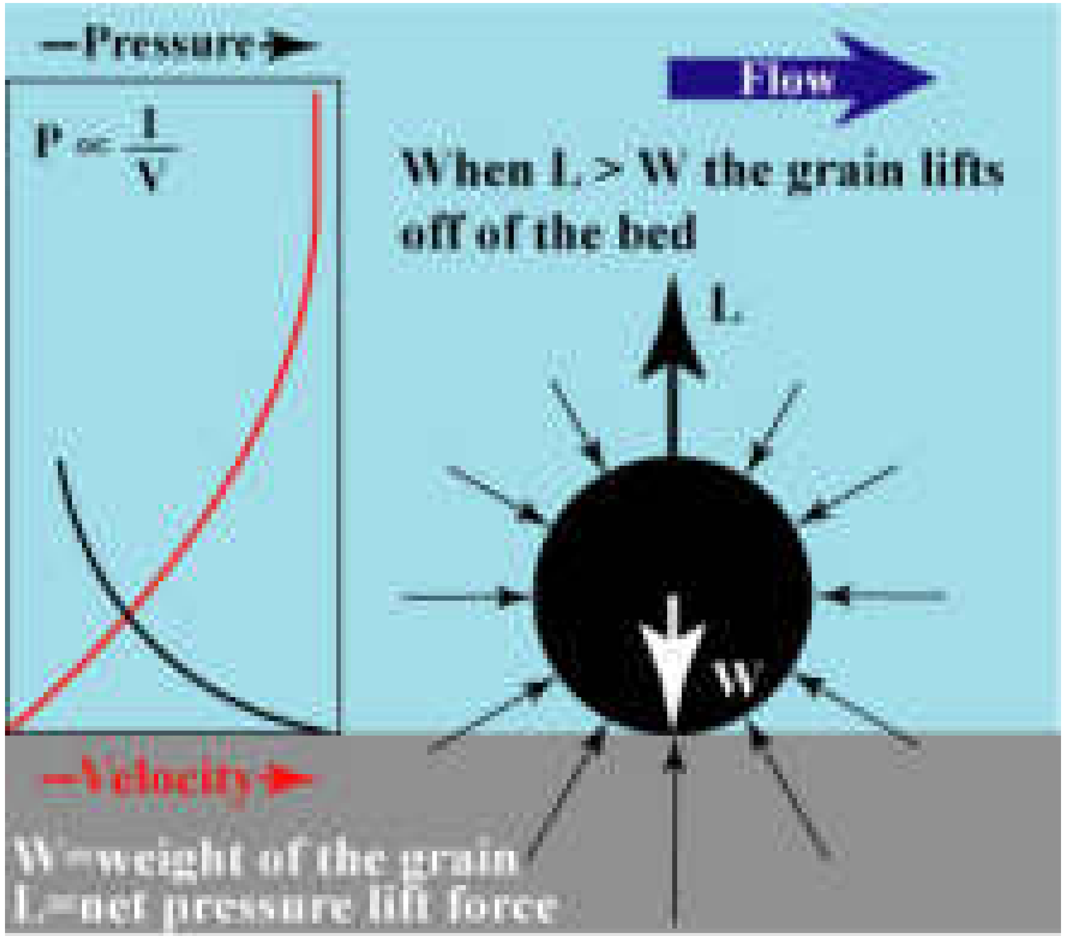


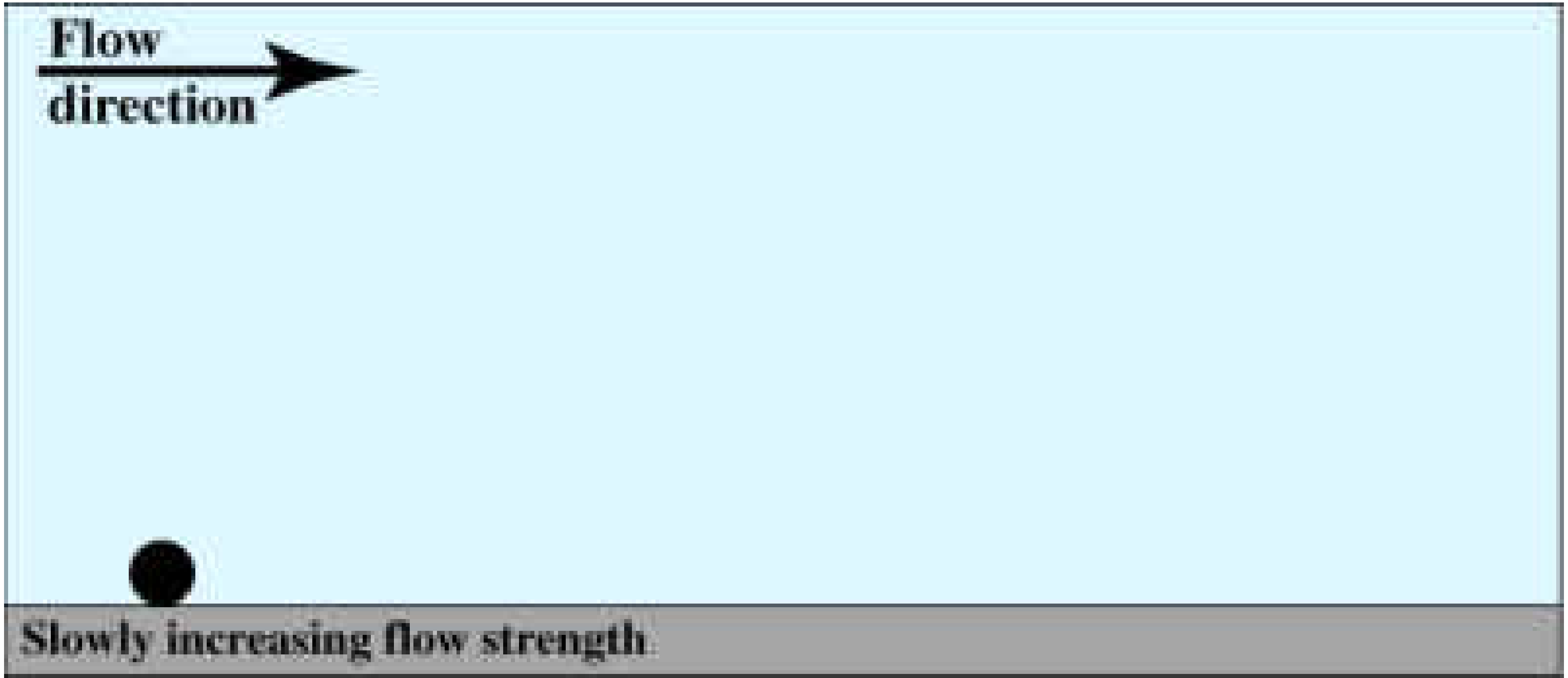
A quick note on saltation……

If the particle remains immobile to the flow and the velocity gradient is large enough so that the Lift force exceeds the particle's weight….it will jump straight upwards away from the bed.

Once off the bed, the pressure difference from top to bottom of the particle is lost and it is carried down current as it falls back to the bed….

following the ballistic trajectory of saltation.




Flow
direction
Slowly increasing flow strength

Shield’s experiments involved determining the critical boundary shear stress required to move spherical particles of various size and density over a bed of grains with the same properties (uniform spheres).

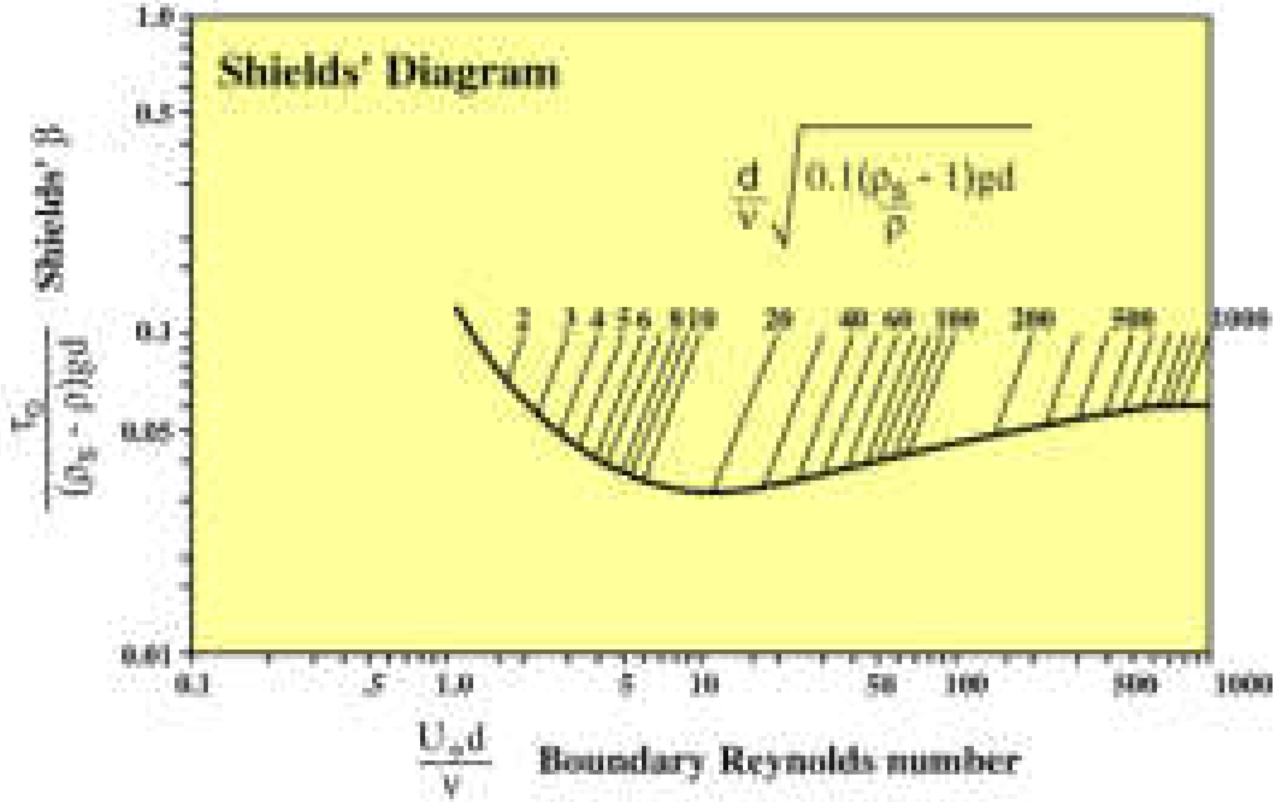
He produced a diagram that allows the determination of the critical shear stress required for the initiation of motion.

A bivariate plot of “Shield’s Beta” versus Boundary Reynolds’ Number:

$$\beta = \frac{\tau_o}{(\rho_s - \rho)gd}$$

vs

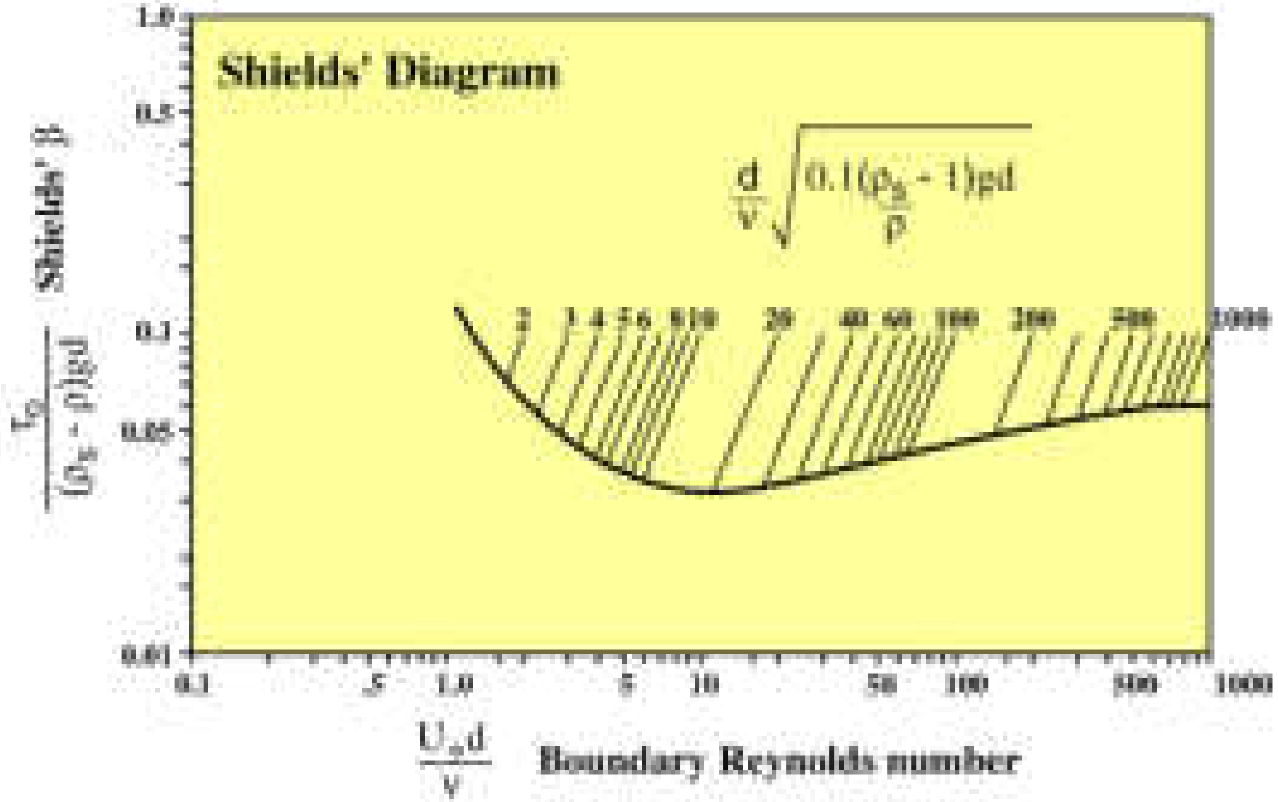
$$R_* = \frac{U_* d}{\upsilon}$$



Critical shear stress for motion.

$$\beta = \frac{\tau_o}{(\rho_s - \rho) g d}$$

Submerged weight of grains per unit area on the bed.

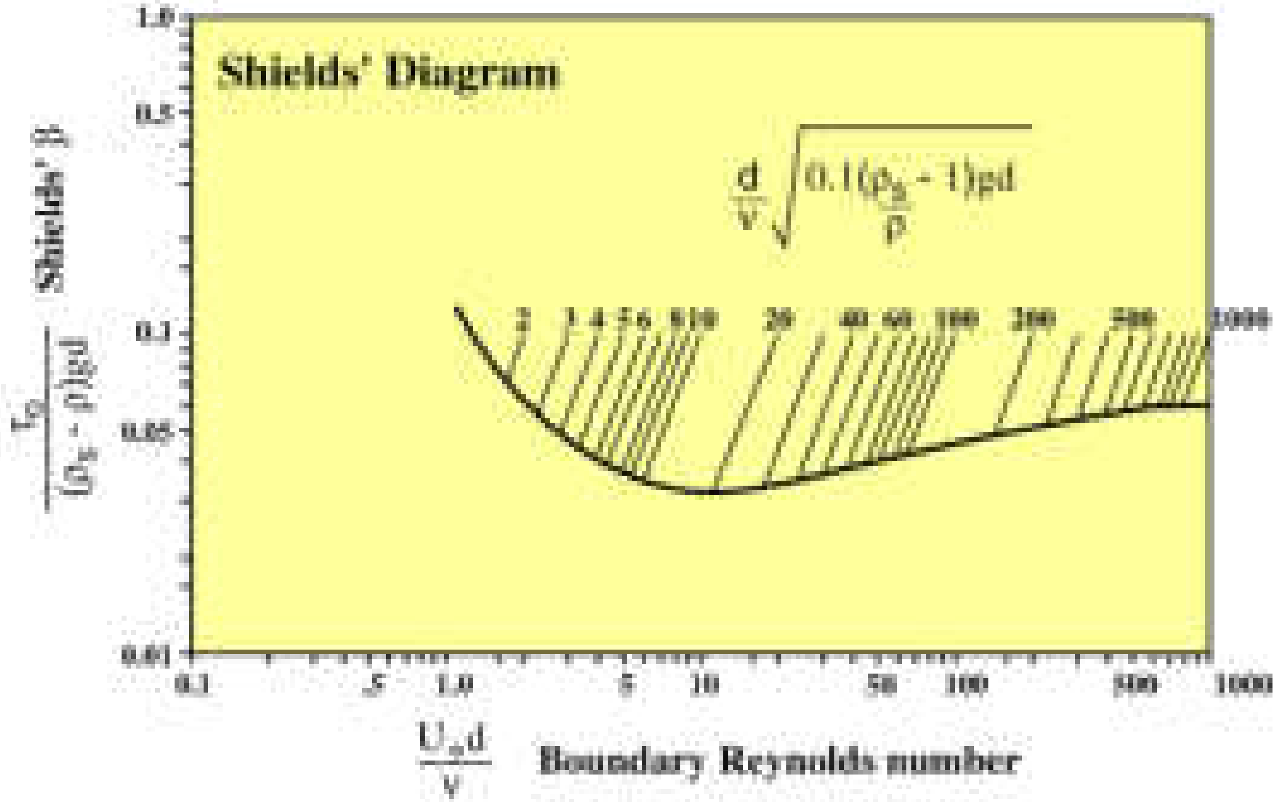


$$\beta = \frac{\tau_o}{(\rho_s - \rho) g d} = \frac{\text{Force acting to move the particle (excluding Lift)}}{\text{Force resisting movement}}$$

As the Lift Force increases β will decrease (lower τ_o required for movement).

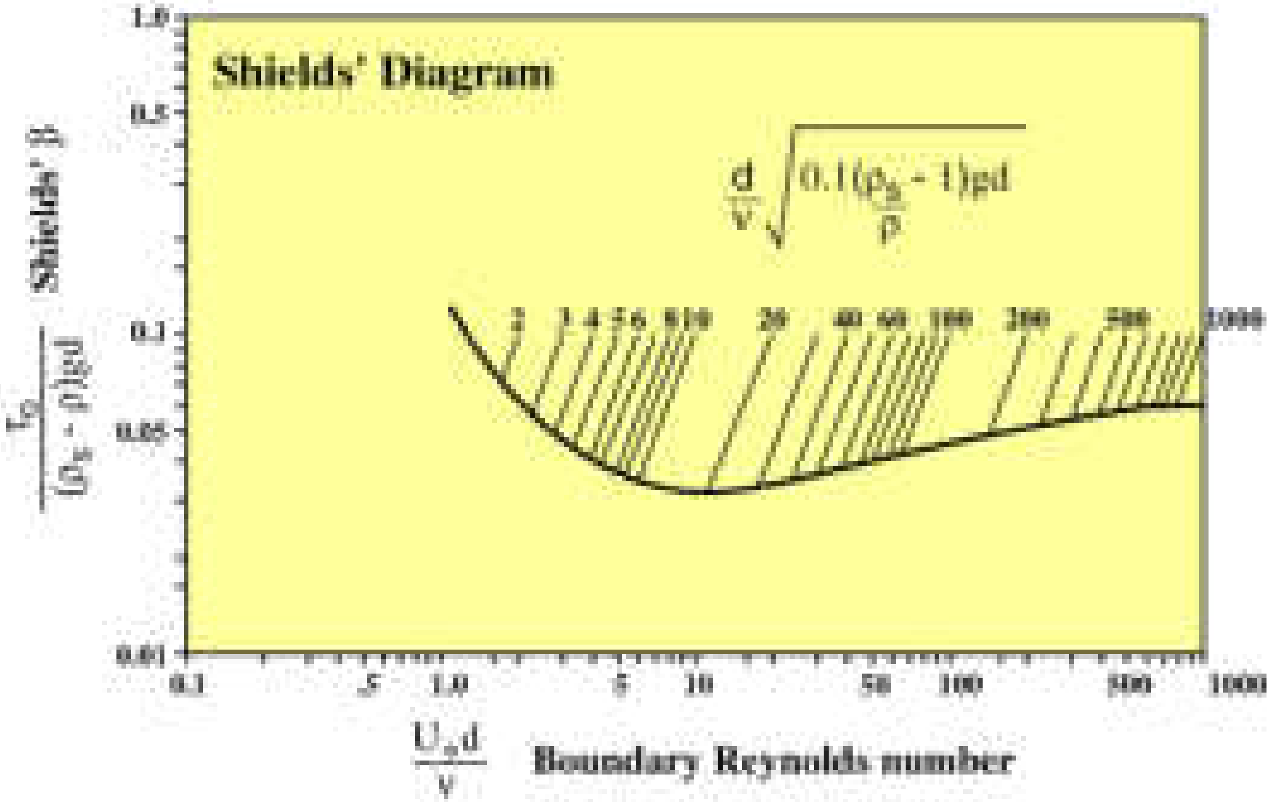
$$R_* = \frac{U_* d}{\upsilon}$$

Reflects something of the lift force (related to the velocity gradient across the particle).



For low boundary Reynold's numbers Shield's β decreases with increasing R_*.

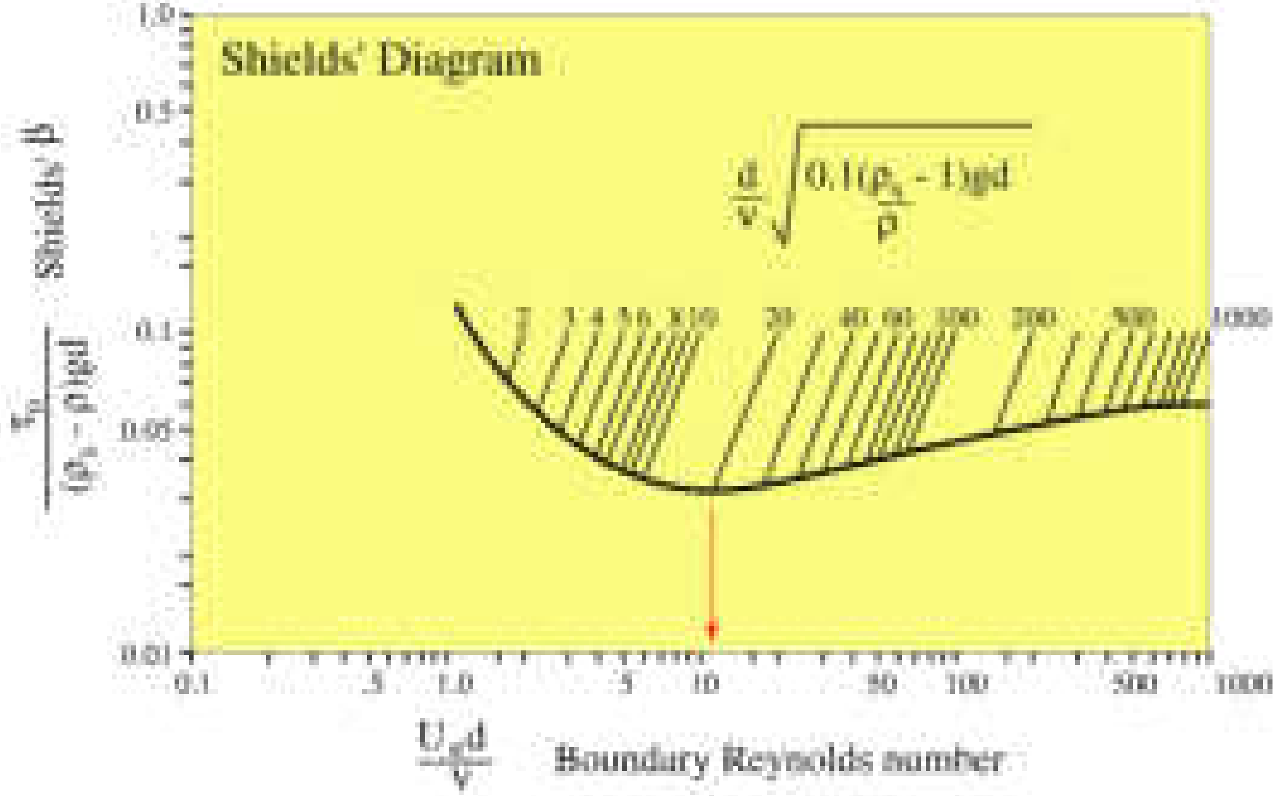
For high boundary Reynold's numbers Shield's β increases with increasing R_*.



For low boundary Reynold's numbers Shield's β decreases with increasing R_*.

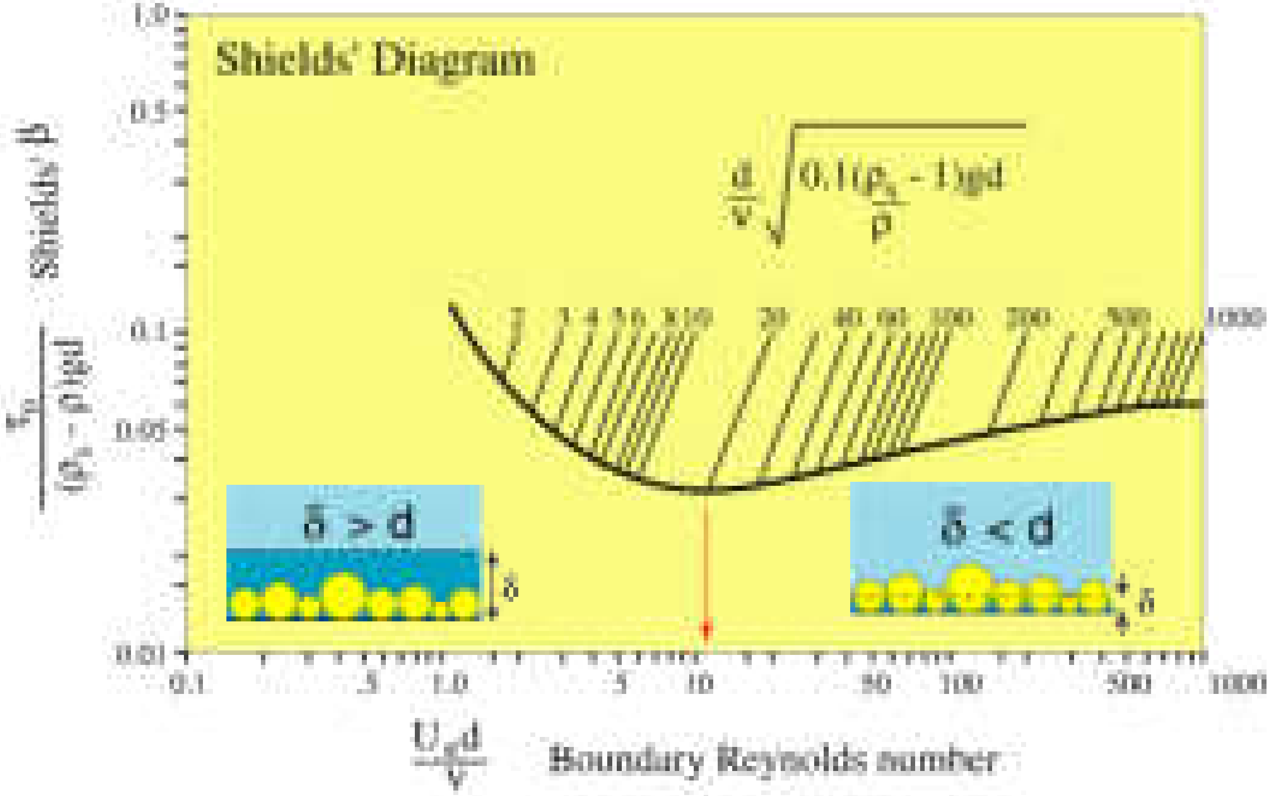
For high boundary Reynold's numbers Shield's β increases with increasing R_*.

The change takes place at $R^* \approx 12$.

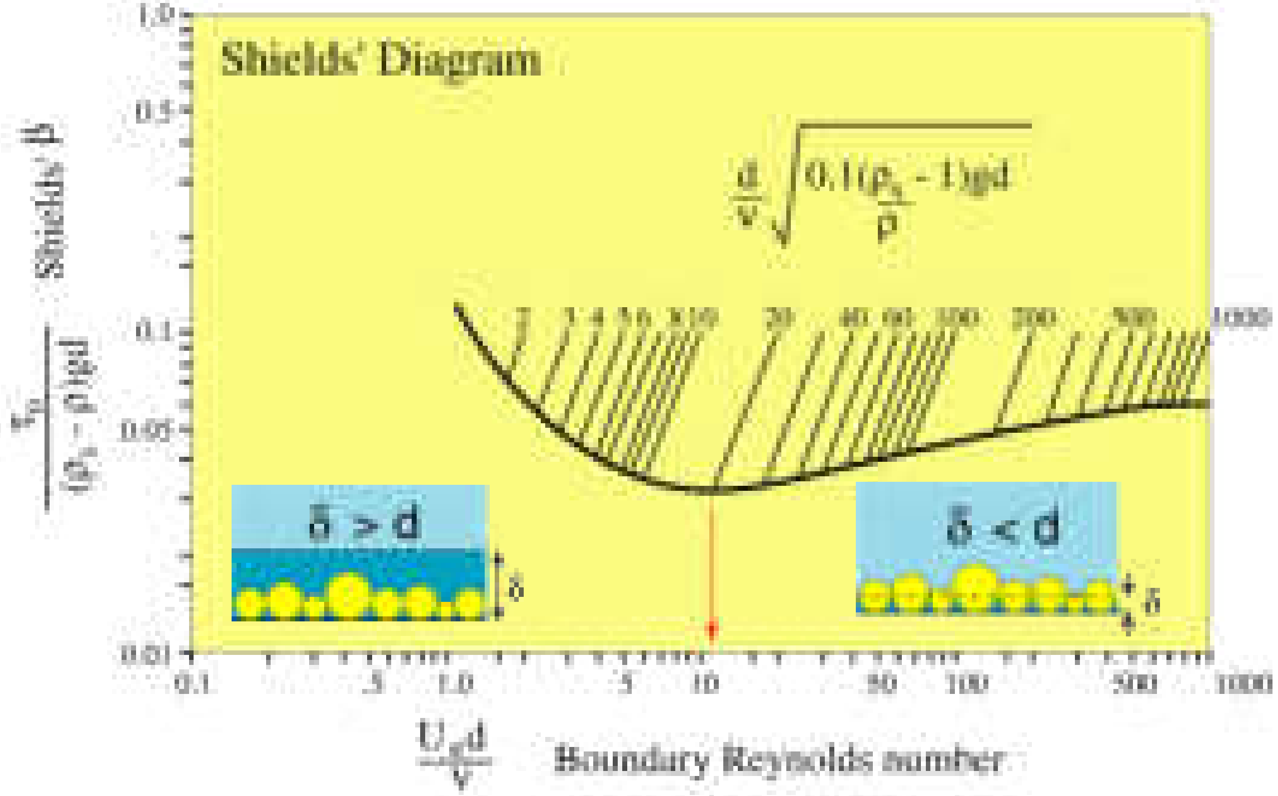


At boundary Reynold's numbers less than 12 the grains on the bed are entirely within the viscous sublayer.

At boundary Reynold's numbers greater than 12 the grains on the bed extend above the viscous sublayer.



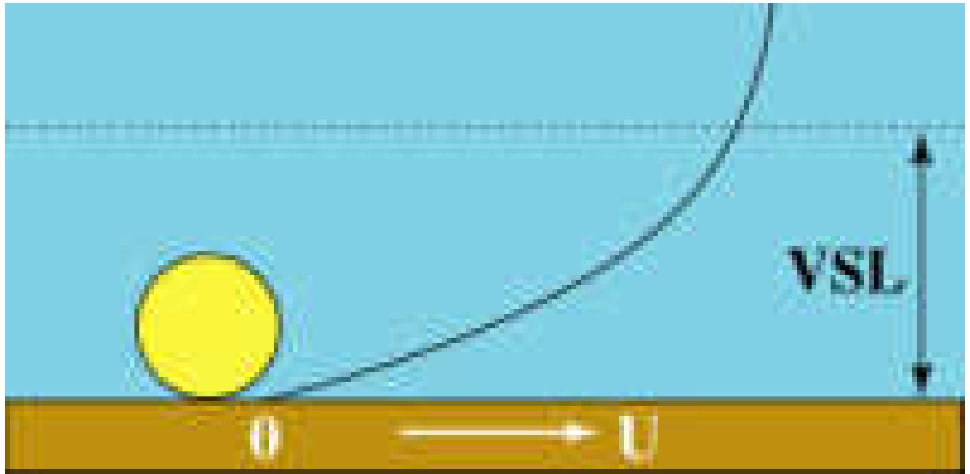
As Shield's β decreases ($R_* < 12$) the critical shear stress required for motion decreases for a given grain size.



At low boundary Reynolds numbers (< 12) the grains experience a strong velocity gradient within the VSL.

As R_* increases towards a value of 12 the VSL thins and the velocity gradient becomes steeper, increasing the lift force acting on the grains.

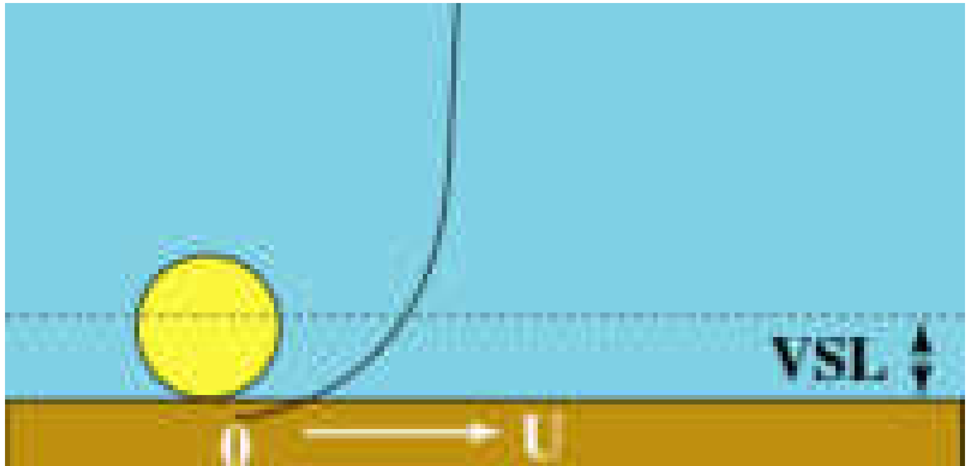
The greater lift force reduces the effective weight of the grains and reduces the boundary shear stress that is necessary to move the grain.



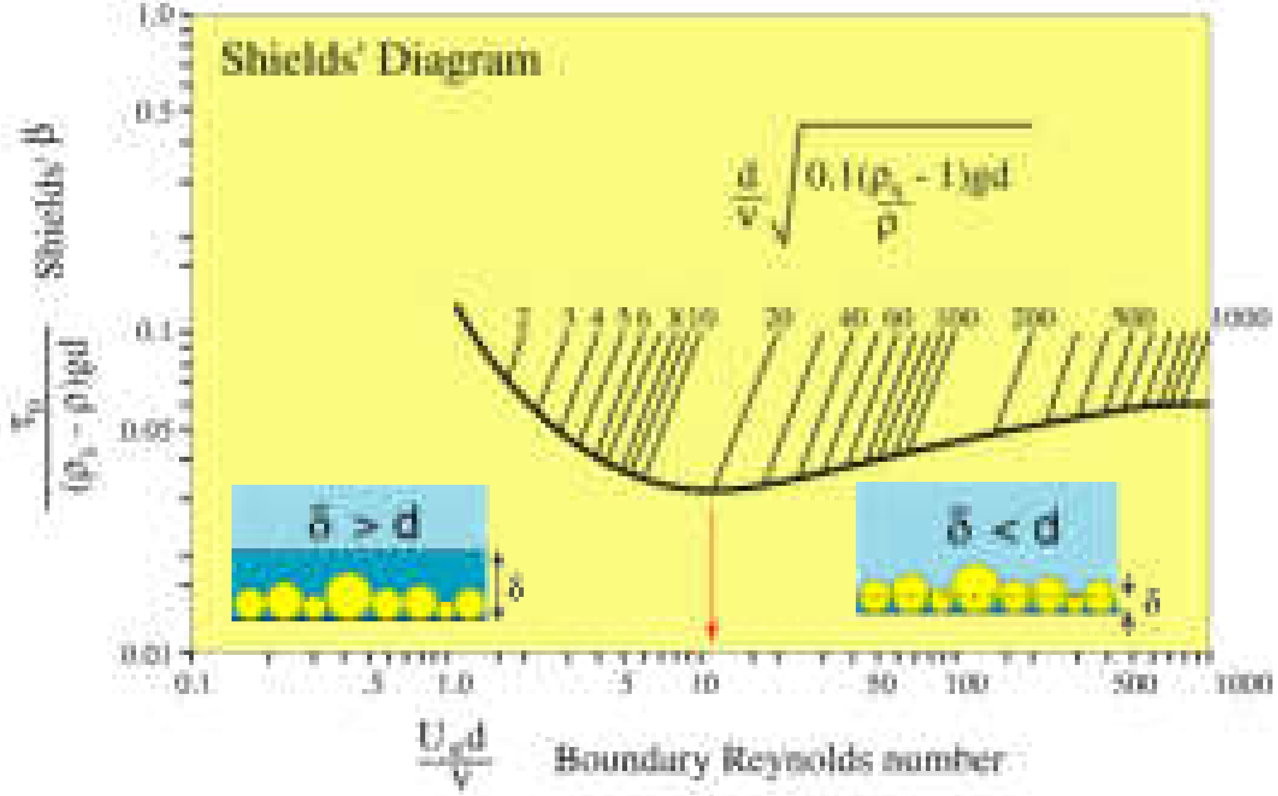
At high boundary Reynolds numbers (> 12) the grains protrude through the VSL so that the region of strong velocity gradient is below the grains, leading to lower lift forces.

As R_* increases the velocity gradient acting on the grains is reduced and resulting lift forces are reduced.

The lower lift force leads to an increase in the effective weight of the grains and increases the boundary shear stress that is necessary to move the grains.

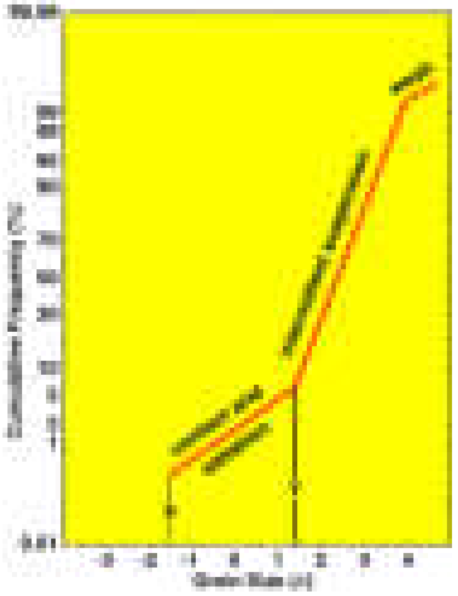


The boundary Reynold’s number accounts for the variation in lift force on the grains which influences the critical shear stress required for motion.



How to use Shield's Diagram

What is the boundary shear stress required to move 2.18 mm sand?



$d = -1.5\phi = 2.8$ mm = 0.0028 m

$\nu = 1.1 \times 10^{-6}$ m^2/s (water at 20°C)

$\rho = 998.2$ kg/m^3 (density of water at 20°C)

$\rho_s = 2650$ kg/m^3 (density of quartz)

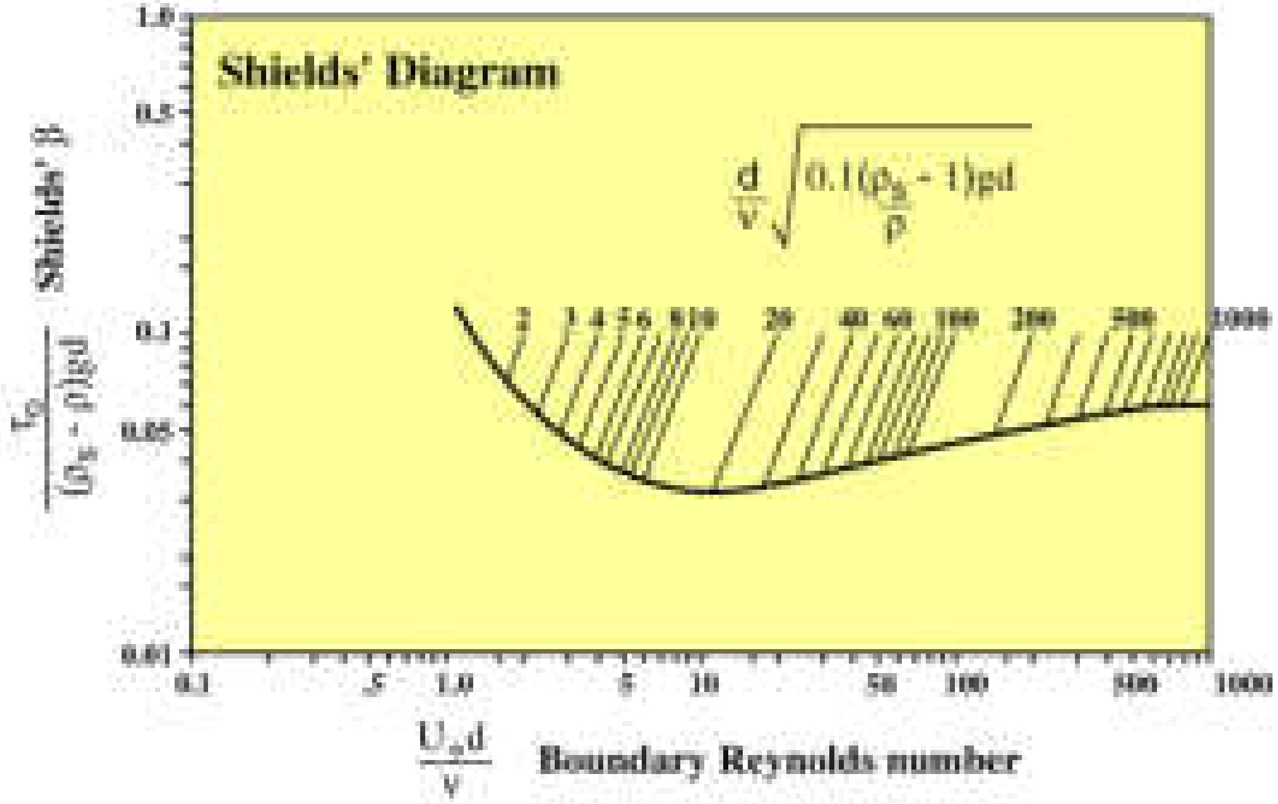
$g = 9.806$ m/s^2

Note the assumptions regarding the water

Calculate:

$$\frac{d}{\upsilon}\sqrt{0.1\left(\frac{\rho_s}{\rho}-1\right)gd}$$

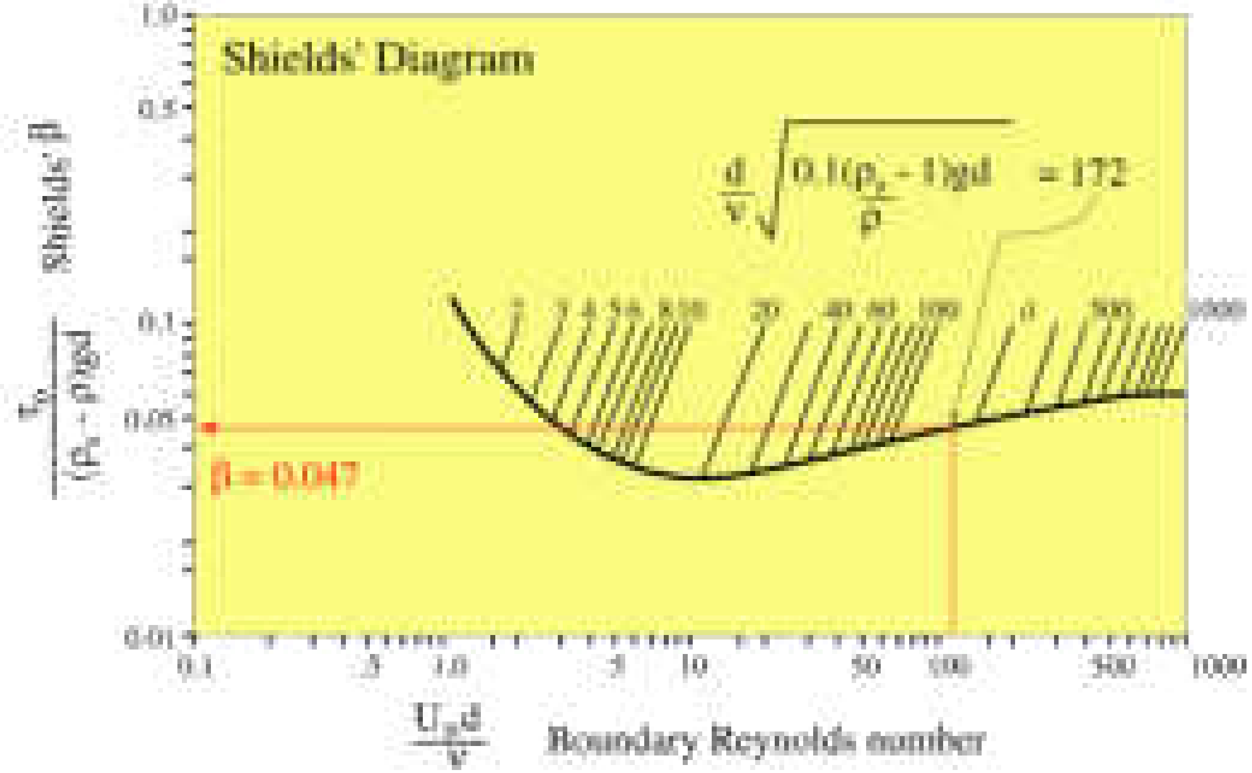
=172



$$\beta = \frac{\tau_o}{(\rho_s - \rho) g d} = 0.047$$

Rearranging:

$$\tau_o = \beta(\rho_s - \rho) g d$$
$$= 0.047(\rho_s - \rho) g d$$
$$= 2.13 \text{ N/m}^2$$



Limitations of Shield's Criterion:

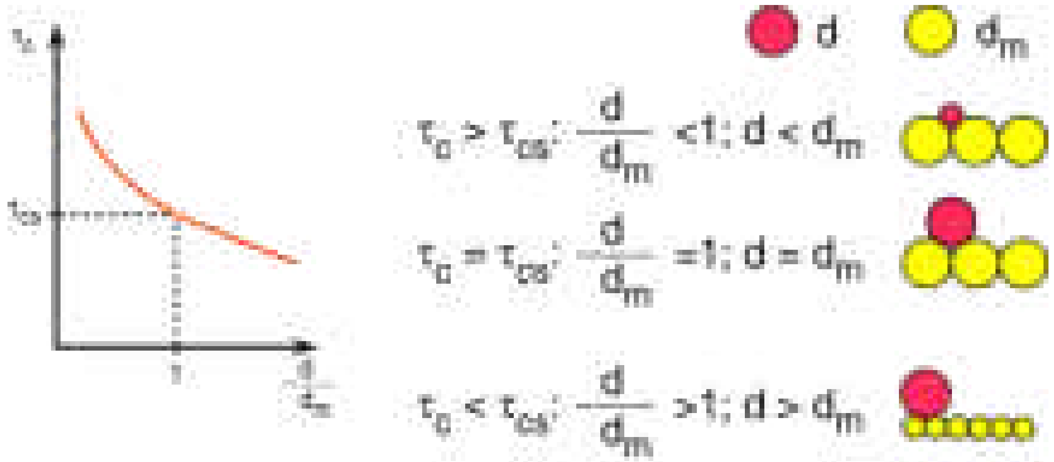
1. It applies only to spherical particles; it doesn't include the influence of particle shape.

It will underestimate the critical shear stress required for motion for angular grains.

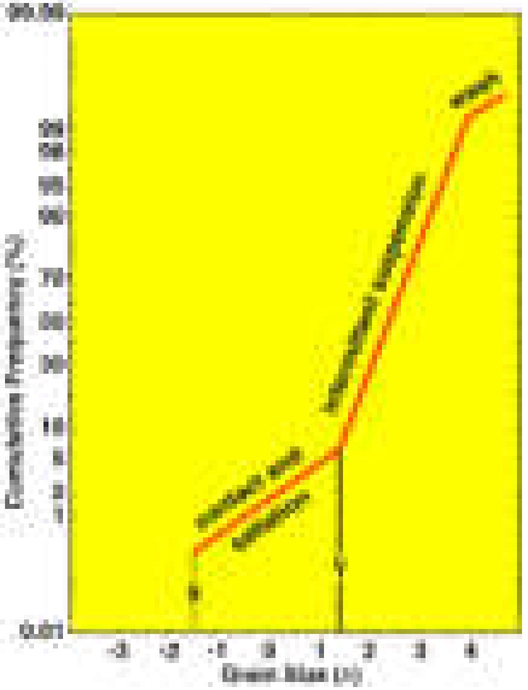
2. It assumes that the material on the bed is of uniform size.

It underestimates the critical shear stress for small grains on a bed of larger grains

It overestimates the critical shear stress for large grains on a bed of finer grains



b) The threshold for suspension

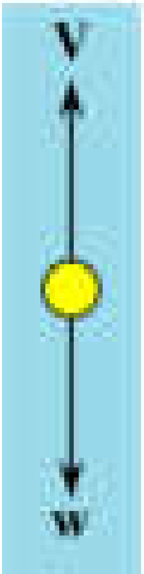


The coarsest grain size in the intermittent suspension load is the coarsest sand that the current will suspend.

Sediment is suspended by the upward component of turbulence (velocity V).

Middleton's criterion for suspension:

The largest particle to be suspended by a current will be that particle with a settling velocity (ω) that is equal to V.



Suspension when $V \geq \omega$

where V is the upward component of velocity due to turbulence and ω is the settling velocity of the particle.

Experiments have shown that $V \approx U_*$ for a given current.

Therefore, Middleton's criterion is:

A particle will be taken into suspension by a current when the shear velocity of the current equals or exceeds the settling velocity of the particle.

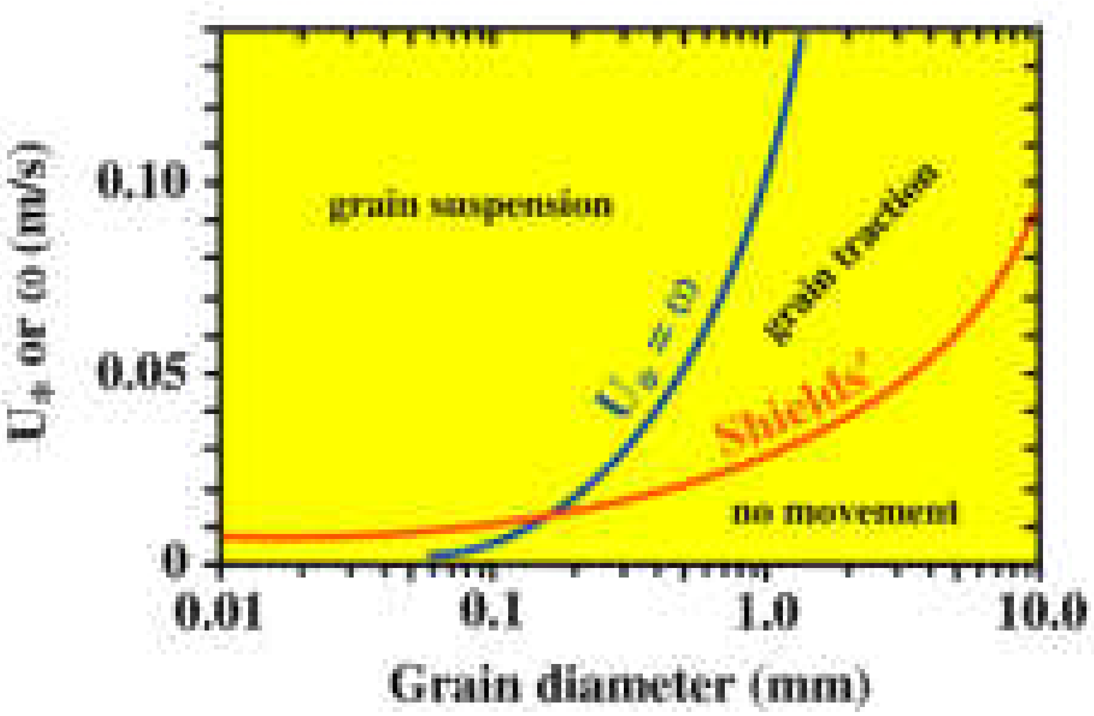
$$U^* \geq \omega$$

Comparisons of the settling velocity of the largest grain size in the intermittent suspension load found in the bed material of major rivers show that they compare very favorably to the measured shear velocity during peak flow in those rivers.

River	**U*** **(m/s)**	**ω** **(m/s)**
Middle Loup	7 – 9	7 - 9
Middle Loup	≈ 20	≈20
Niobrara	7 - 10	7 - 9
Elkhorn	7 - 9	2.5 – 5.0
Mississippi (Omaha)	6.5 – 6.8	2.5 – 5.0
Mississippi (St. Louis)	9 - 11	3 - 12
Rio Grande	8 - 12	≈10

This diagram shows the shear velocity required to suspend particles as a function of their size (the curve labeled $U_* = \omega$).

For comparison it also shows the critical shear velocity required to move a particle on the bed based on Shield's criterion.

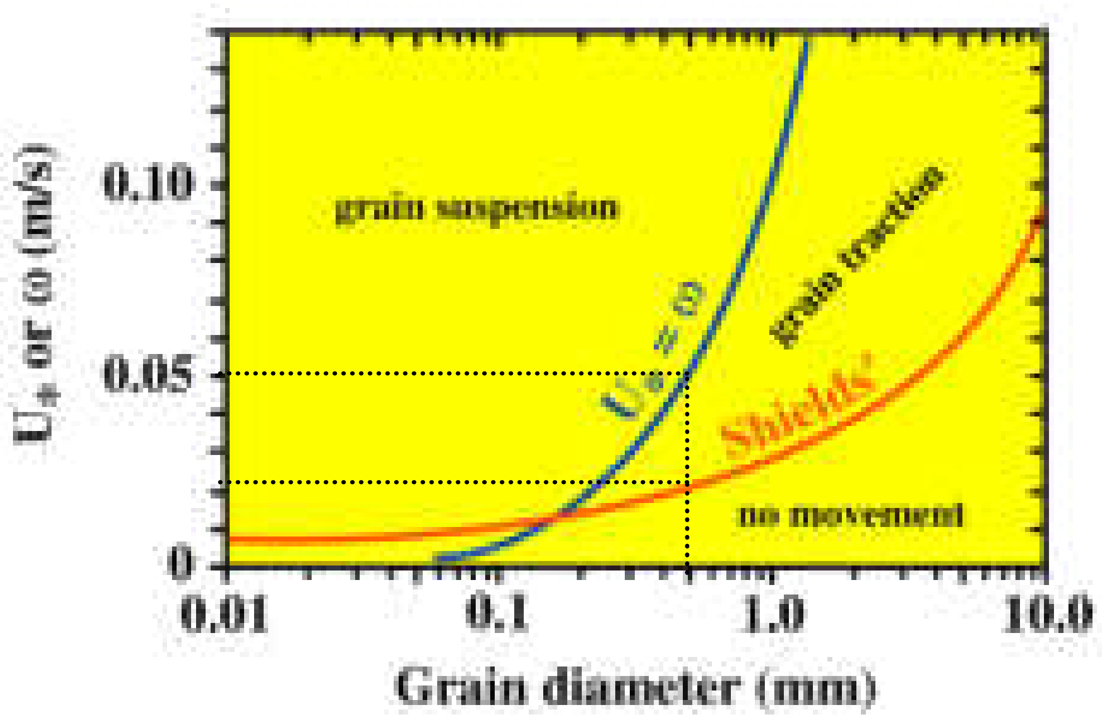


The $U_* = \omega$ curve can also be used to estimate settling velocity for grains coarser than 0.1 mm (the upper limit for Stoke's Law).

For 0.5 mm diameter quartz spheres:

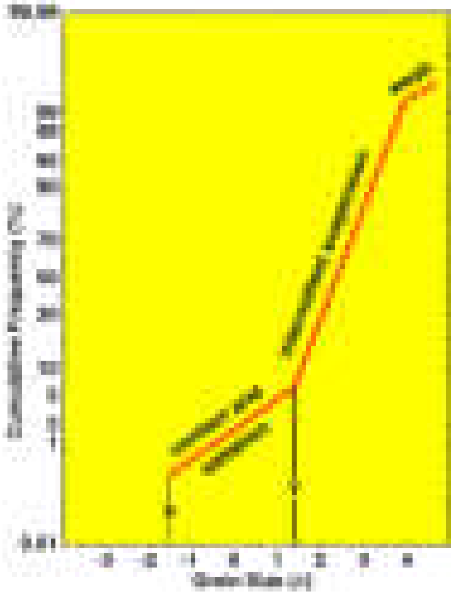
As flow strength increases at U*=0.021 m/s the grain will begin to move on the bed.

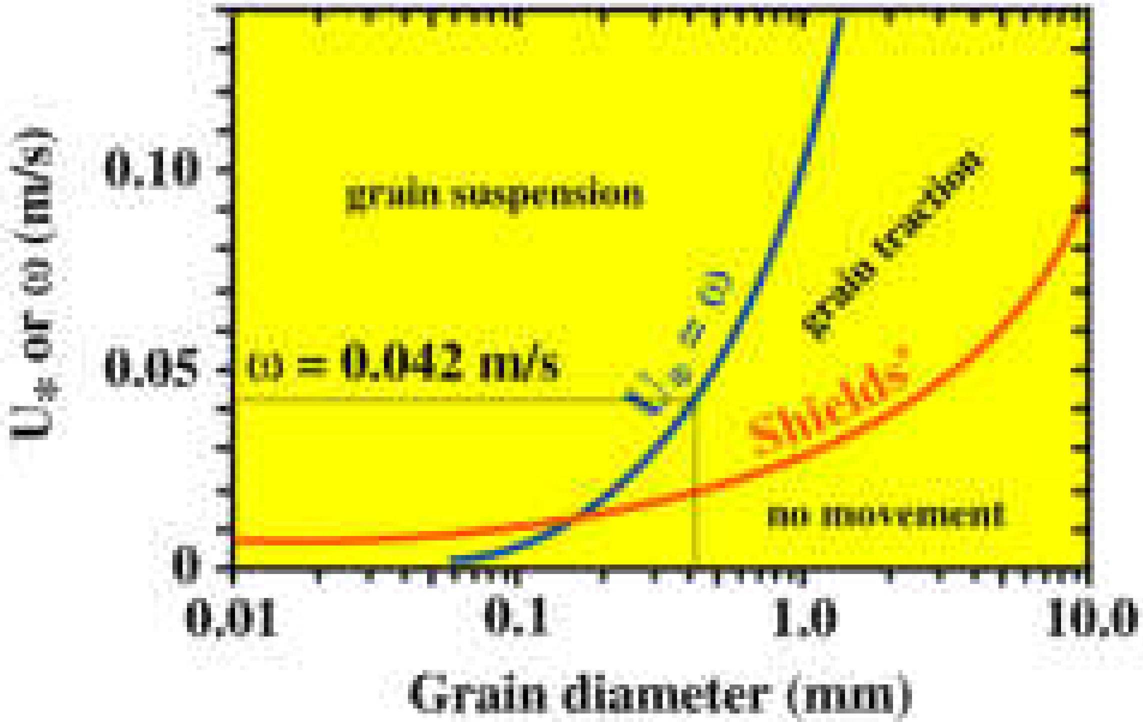
As flow strength increases further at U*=0.05 m/s the moving grain will be taken into suspension.



Note that for grain sizes finer than approximately 0.015 mm the grains will go into suspension as soon as the flow strength is great enough to move them (i.e., they will not move as contact load).

What is the critical shear velocity required to suspend 0.41 mm sand?





The critical shear velocity for suspension is 0.042 m/s.

How do our estimates based on the coarsest grains size in transport on the bed and the coarsest grain size in suspension compare?

Middleton's criterion: $U_* = 0.042$ m/s

Shield's criterion: $\tau_o = 2.13$ N/m^2

$$U_* = \sqrt{\frac{\tau_o}{\rho}}$$

$\rho = 998.2$ kg/m^3 (density of water at 20°C)

$U_* = 0.046$ m/s

Very close!

Contemporaneously, Ekman…

V.W. Ekman

Young Ekman

Considered the effects of rotation although he did not really think of his solutions in terms of what we would call boundary layer theory.

This is a singular perturbation.

The order of the equations is reduced and we can no longer satisfy all the boundary conditions if the viscous term is neglected.

The mathematical issue is how to retain the higher order derivatives only where they are needed to help satisfy the boundary conditions and

the physical issue is to understand through the applications of boundary layer theory how (and whether) the action of friction in very localized regions may affect the fluid flow in regions *outside* the domain directly affected by friction. The interplay between the *outer* region, in which friction is not directly important, and the *inner* region in which friction directly acts is a key feature of boundary layer theory (a form of *singular perturbation theory*).

An Oceanic example
Wind-driven ocean circulation model

$$\varepsilon J(\psi,\nabla^2\psi)+\psi_x=-r\nabla^2\psi+\nu\nabla^4\psi+T(x,y)$$

$$J(a,b)\equiv a_x b_y - a_y b_x \qquad \varepsilon = U/\beta L^2$$

$$\nu=\frac{A_H}{\beta L^3},\quad r=\frac{r_*}{\beta L}$$

If r and ν neglected and the no slip condition is dropped, there will still be a singular perturbation to the equations if the ε term, i.e. the nonlinear advection terms are ignored. This leads to an *inertial* boundary layer.This equation in its entirety will be discussed more fully later.

An outline of where we will be going

1) Linear boundary layer theory

Ekman layers, Boundary layers in density stratified fluids, control of interior, experimental applications.

2) Coastal bottom boundary layer.

Boundary layer on shelf for upwelling and downwelling.

Observations (Lentz)

3) Boundary layers in the General Oceanic Circulation.

Sverdrup theory, Stommel, Munk, inertial boundary layers, inertial runaway,thermocline and its boundary layer structure.

Equations of motion

$$uu_x + vu_y + wu_z - 2\Omega v = -\frac{1}{\rho} p_x + \nu\left[u_{xx} + u_{yy} + u_{zz}\right]$$

$$uv_x + vv_y + wv_z + 2\Omega u = -\frac{1}{\rho} p_y + \nu\left[v_{xx} + v_{yy} + v_{zz}\right]$$

$$uw_x + vw_y + ww_z \qquad = -\frac{1}{\rho} p_z - g + \left[w_{xx} + w_{yy} + w_{zz}\right]$$

$$u_x + v_y + w_z = 0$$

Incompressible fluid in a rotating system.

If the density is not constant must add an ernery equation

We are interested in cases where ν is "small". Must introduce scales.