Introduction to Algorithms

Design and Analysis of Algorithms

- **Analysis:** predict the cost of an algorithm in terms of resources and performance
- **Design:** design algorithms which minimize the cost
- **Algorithm:** An algorithm is a step-by-step procedure for solving a problem in a finite amount of time.

The problem of sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Insertion sort

“pseudocode”

INSERTION-SORT (A, n) $\triangleright$ $A[1 \,.\,.\, n]$

for $j \leftarrow 2$ **to** n

  do $key \leftarrow A[j]$

    $i \leftarrow j - 1$

    while $i > 0$ and $A[i] > key$

      do $A[i+1] \leftarrow A[i]$

        $i \leftarrow i - 1$

    $A[i+1] = key$

n

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 $\quad$ 2 $\quad$ 4 $\quad$ 9 $\quad$ 3 $\quad$ 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6

Example of insertion sort

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Example of insertion sort

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Example of insertion sort

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Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*

Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(n^2\right) \qquad \text{[arithmetic series]}$$

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta\left(n^2\right)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Major Simplifying Convention: Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
  - $T_A(n)$ = time of A on length n inputs
- Generally, we seek upper bounds on the running time, to have a guarantee of performance.

Kinds of analysis

Worst-case: (usually)

$T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

$T(n)$ = expected time of algorithm over all inputs of size n.

- Need assumption of statistical distribution of inputs.

Best-case: (NEVER)

- Cheat with a slow algorithm that works fast on *some* input.

Θ-notation

DEF:

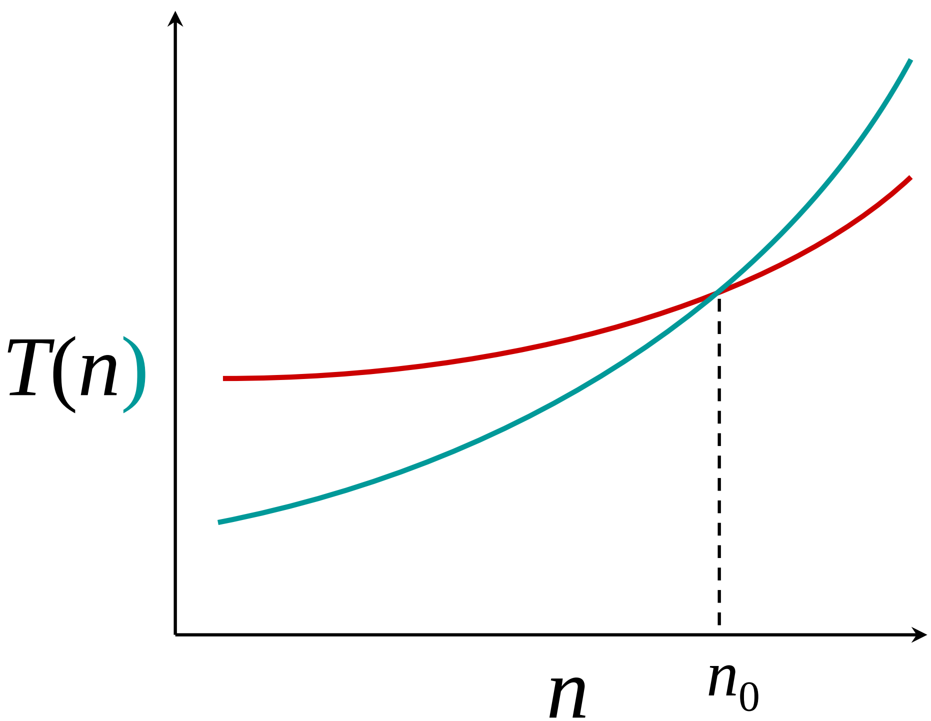
$\Theta(g(n)) = \{ f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

Basic manipulations:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- Asymptotic analysis is a useful tool to help to structure our thinking toward better algorithm
- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing

Insertion sort analysis

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Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $$\boldsymbol{f(n) \le cg(n)} \quad \text{for } \boldsymbol{n \ge n_0}$$

- Example: $2\boldsymbol{n} + 10$ is $\boldsymbol{O(n)}$
  - $2\boldsymbol{n} + 10 \le \boldsymbol{cn}$
  - $(\boldsymbol{c} - 2)\, \boldsymbol{n} \ge 10$
  - $\boldsymbol{n} \ge 10/(\boldsymbol{c} - 2)$
  - Pick $\boldsymbol{c} = 3$ and $\boldsymbol{n_0} = 10$

Big-Oh Example

- Example: the function $\boldsymbol{n}^2$ is not $\boldsymbol{O}(\boldsymbol{n})$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since $\boldsymbol{c}$ must be a constant

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $7n - 1$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Other Notations

big-Omega

- f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

big-Theta

- f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n)$ for $n \geq n_0$

little-oh

- f(n) is $o(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$

little-omega

- f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$