Linear Control system

Introductory concepts

System – An interconnection of elements and devices for a desired purpose.

Control System – An interconnection of components forming a system configuration that will provide a desired response.

Process – The device, plant, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.

Process to be controlled.

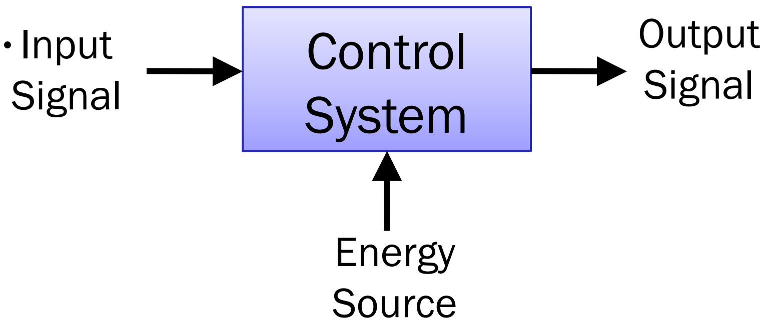
- Plant:It is a process/body/machine of which a entire particular quantity or condition is to be controlled.
- Controller: It is a component required to generate the appropriate control signal applied to a plant.
- Servomechanism:It is a power amplifying feedback control system in which the control variable is mechanical position ,or time derivative of position such as velocity or acceleration .Occasionally it refers to mechanical syatem in which steady state error is zero for constant input signal.

- Regulator:It is a system in which there is steady state value for constant input signal.
- Linear time invariant system:systems whose parameters are varying with time.not dependent on whether input/output are varying with time.
- Non Linear time invariant system:systems whose parameters are not varying with time. Dependent on whether input/output are varying with time.

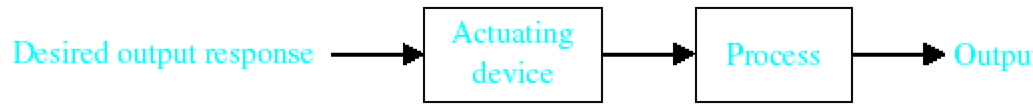
- Open loop system:A system in which the output is dependent on I/p but I/p is independent of change in o/p of the system.
- Closed loop system:A system in which the I/p, controlling action is dependent of change in o/p of the system.
- Continuous and Discrete time control system:All system variables are function of continuous time variable ‘t’.
- Discrete time control system: One or more system variable are known at only at discrete time intervals.

Control System

- Control is the process of causing a system variable to conform to some desired value.
- Manual control → Automatic control (involving machines only).
- A control system is an interconnection of components forming a system configuration that will provide a desired system response.

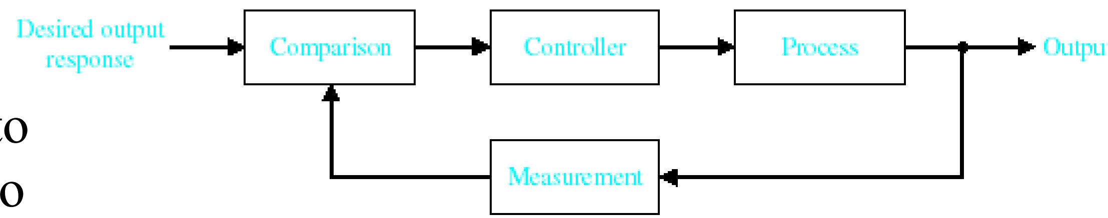


Open-Loop Control Systems utilize a controller or control actuator to obtain the desired response.



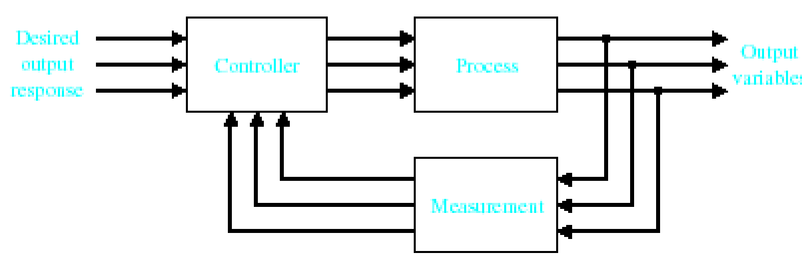
Open-loop control system (without feedback).

Closed-Loop Control Systems utilizes feedback to compare the actual output to the desired output response.



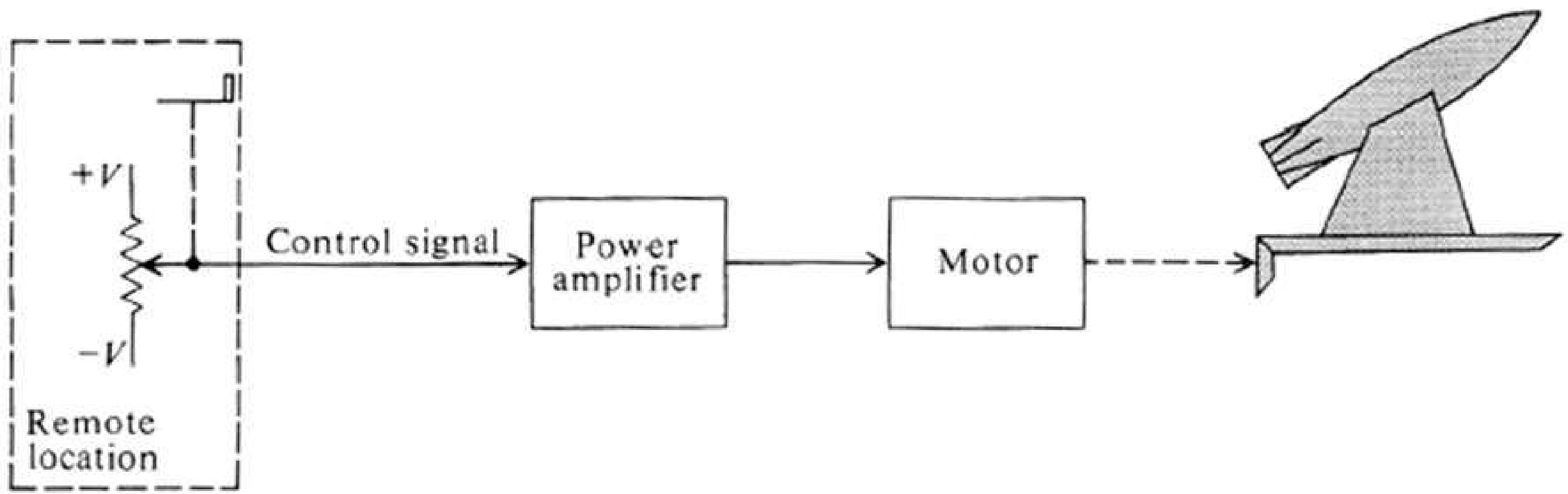
Closed-loop feedback control system (with feedback).

Multivariable Control System



Control System Classification

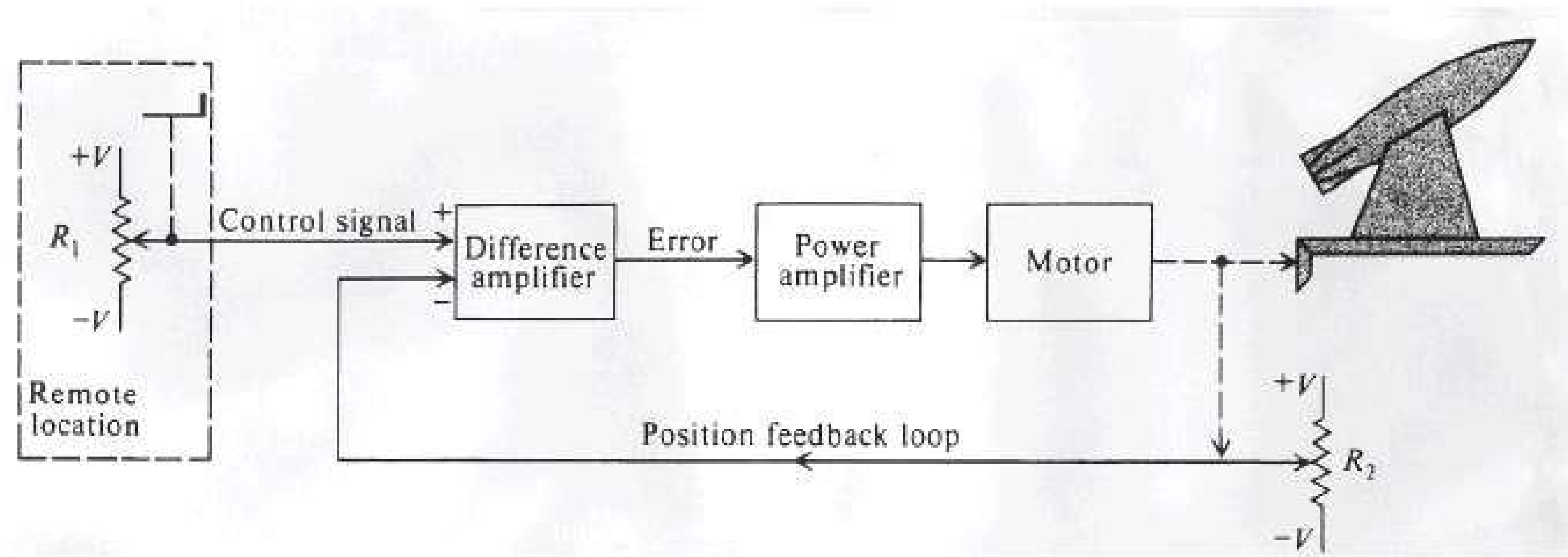
Missile Launcher System



Open-Loop Control System

Control System Classification

Missile Launcher System



Closed-Loop Feedback Control System

Effect of feedback on overall gain.

- The overall gain of a open loop system is $G(s)$ and feedback when introduced the feedback is $C(s)/R(s)=G(s)/1+-G(s)H(s)$.So gain is affected by the denominator.For positive feedback the sign in denominator is(-) and for negative feedback the sign in denominator is(+).

Effect of feedback on overall sensitivity.

- sensitivity is reduced by factor of $1/1+G(s)H(s)$ due to feedback for change in forward parameter.
- Closed loop system is more sensitive to variation in feedback parameters than variation in forward path transfer functions.

Effect of feedback on overall stability.

- feedback can improve o overall stability or may harmful purely depends on application and proper design of feedback.

Effect of feedback on overall noise.

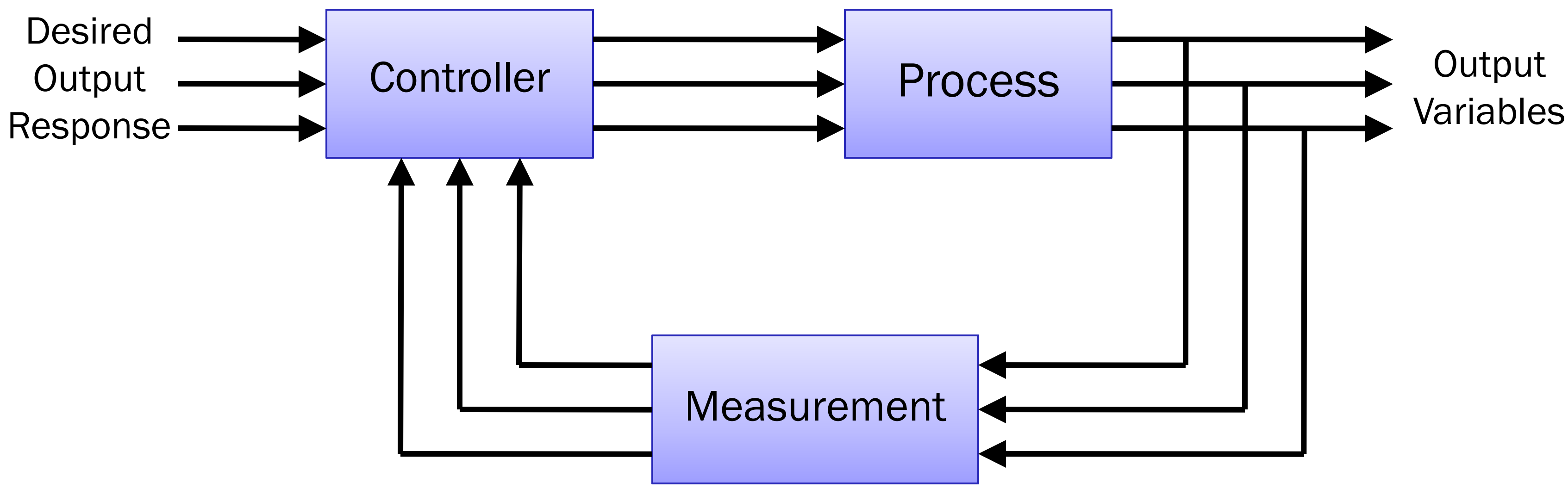
- Feedback can decrease the effect of noise.

Manual Vs Automatic Control

- Control is a process of causing a system variable such as temperature or position to conform to some desired value or trajectory, called reference value or trajectory.
- For example, driving a car implies controlling the vehicle to follow the desired path to arrive safely at a planned destination.
  i. If you are driving the car yourself, you are performing manual control of the car.

  ii. If you use design a machine, or use a computer to do it, then you have built an automatic control system.

Control System Classification



Multi Input Multi Output (MIMO) System

Purpose of Control Systems

i. Power Amplification (Gain)
   - Positioning of a large radar antenna by low-power rotation of a knob

ii. Remote Control
   - Robotic arm used to pick up radioactive materials

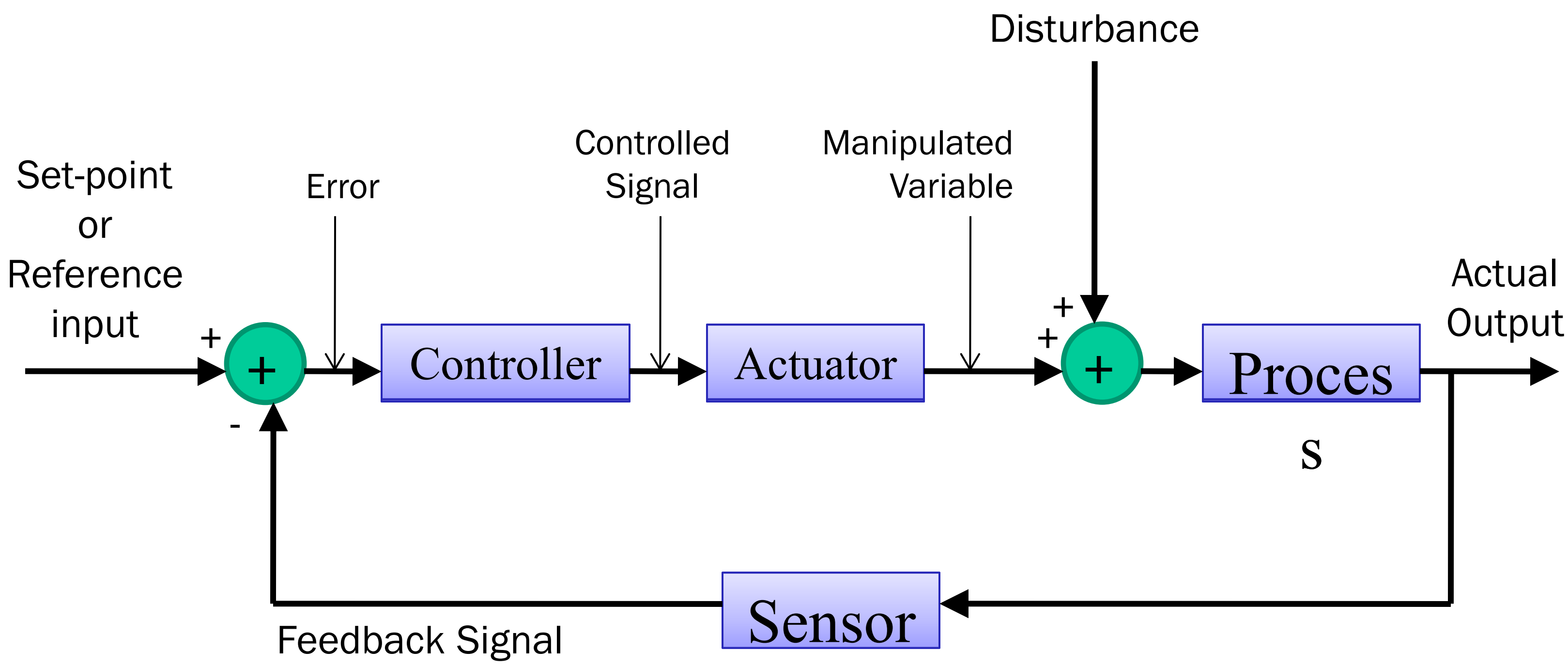
iii. Convenience of Input Form
   - Changing room temperature by thermostat position

iv. Compensation for Disturbances
   - Controlling antenna position in the presence of large wind disturbance torque

Application of control system

- Familiar control systems have the basic closed-loop configuration. For example, a refrigerator has a temperature setting for desired temperature, a thermostat to measure the actual temperature and the error, and a compressor motor for power amplification. Other examples in the home are the oven, furnace, and water heater. In industry, there are controls for speed, process temperature and pressure, position, thickness, composition, and quality, among many others. Feedback control concepts have also been applied to mass transportation, electric power systems, automatic warehousing and inventory control, automatic control of agricultural systems, biomedical experimentation and biological control systems, and social, economic, and political systems. *See also* Biomedical engineering; Electric power systems; Mathematical biology; Systems analysis; Systems engineering.

General Control System



Control System Components

i. System, plant or process
   - To be controlled

ii. Actuators
   - Converts the control signal to a power signal

iii. Sensors
   - Provides measurement of the system output

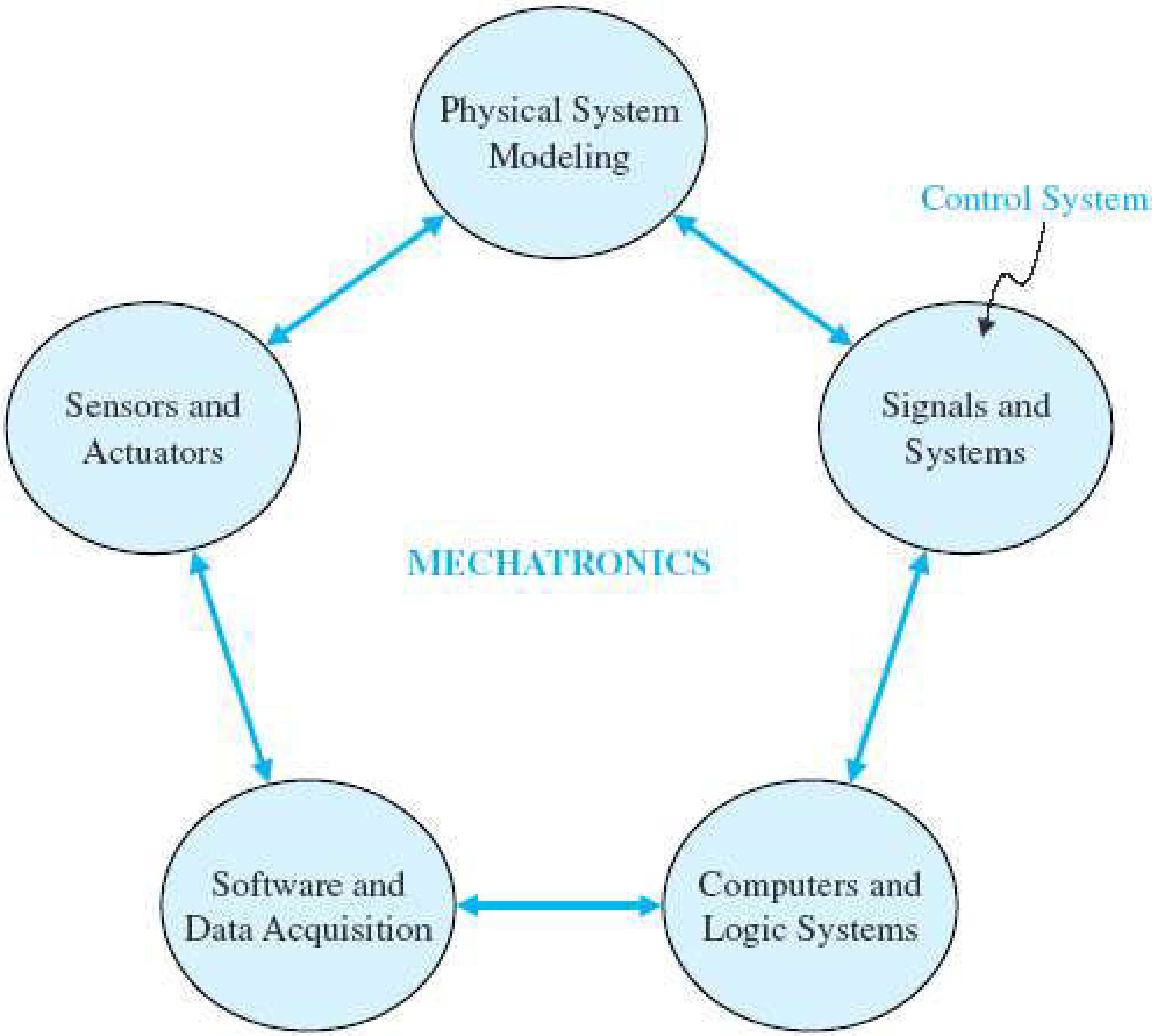
iv. Reference input
   - Represents the desired output

Control System Design Process

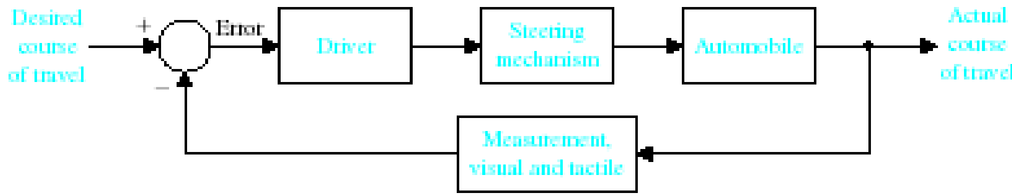
1. Establish control goals
2. Identify the variables to control
3. Write the specifications for the variables
4. Establish the system configuration and identify the actuator
5. Obtain a model of the process, the actuator and the sensor
6. Describe a controller and select key parameters to be adjusted
7. Optimize the parameters and analyze the performance

If the performance does not meet specifications, then iterate the configuration and actuator

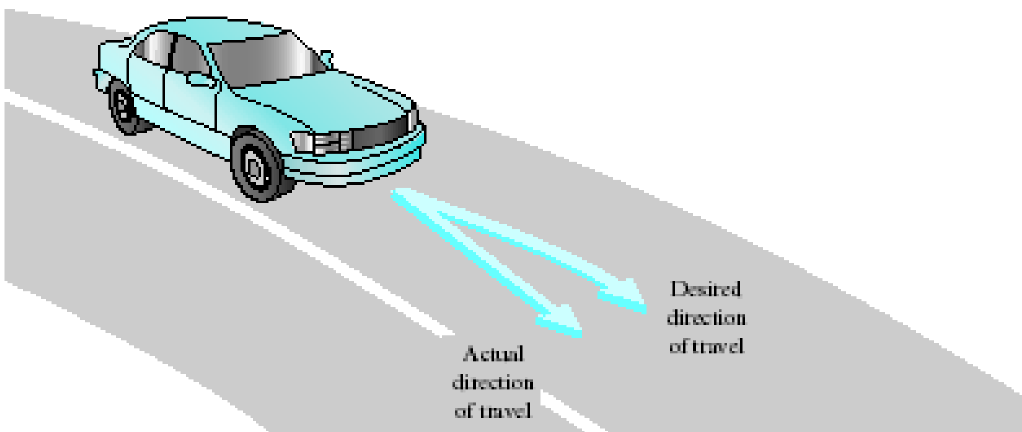
If the performance meet the specifications, then finalize design


Physical System Modeling
Control Systems
Sensors and Actuators
Signals and Systems
MECHATRONICS
Software and Data Acquisition
Computers and Logic Systems

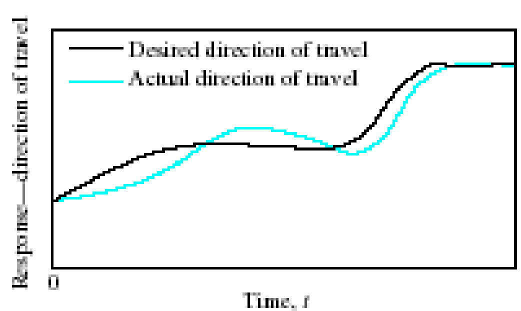
Examples of Modern Control Systems



(a)



(b)



(c)

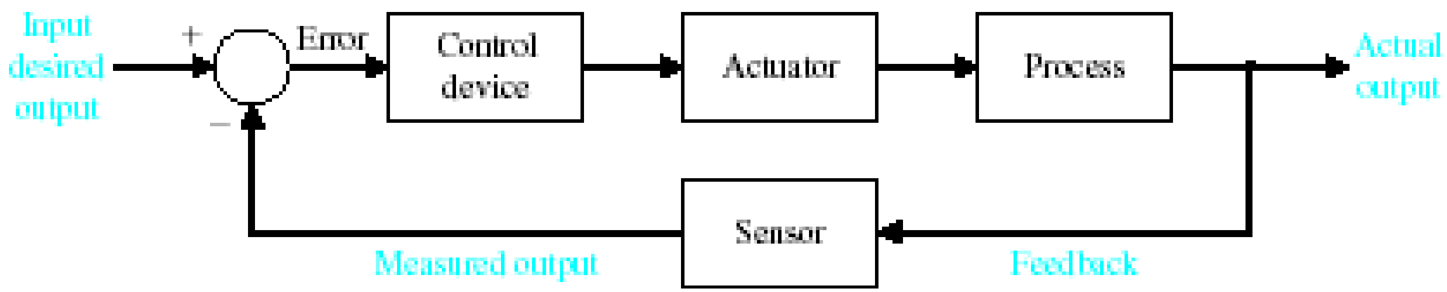
(a) Automobile steering control system.

(b) The driver uses the difference between the actual and the desired direction of travel

to generate a controlled adjustment of the steering wheel.

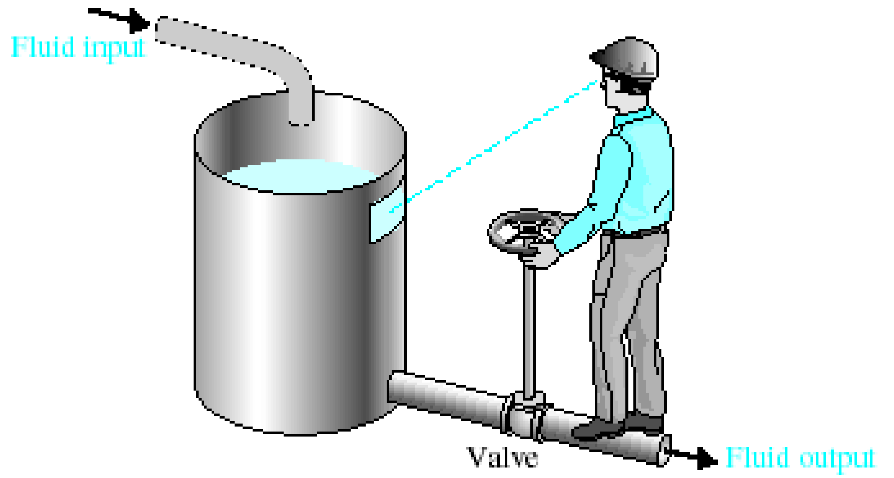
(c) Typical direction-of-travel response.

Examples of Modern Control Systems



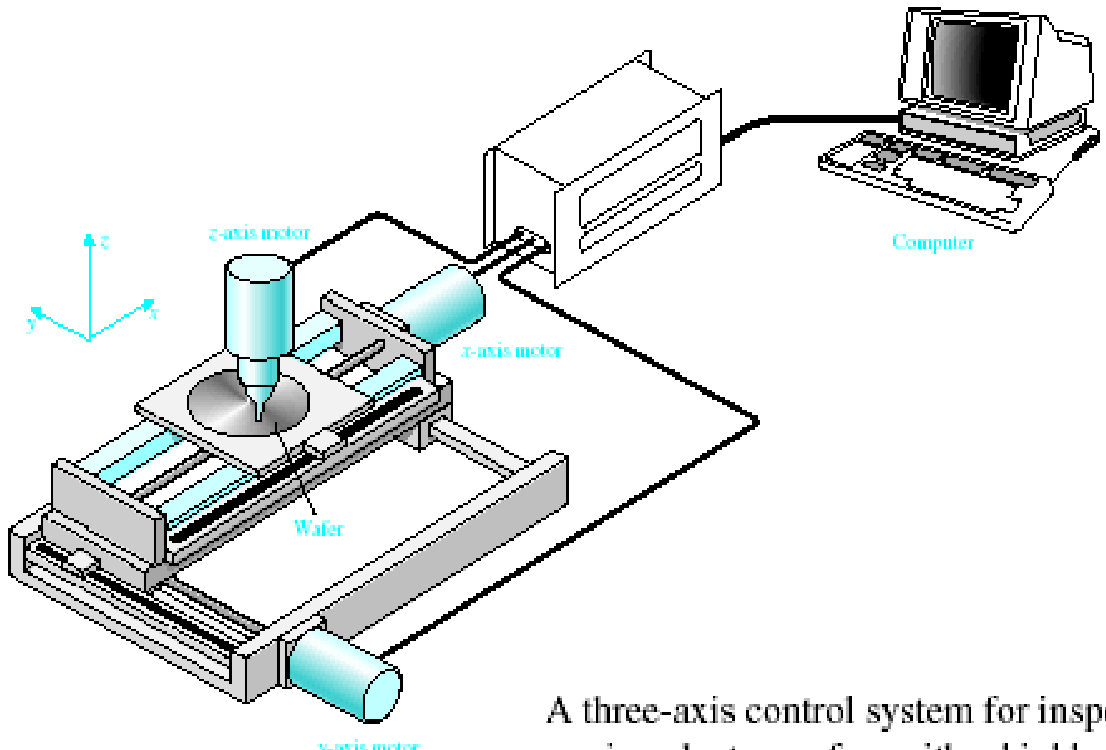
A negative feedback system block diagram depicting a basic closed-loop control system.
The control device is often called a "controller."

Examples of Modern Control Systems



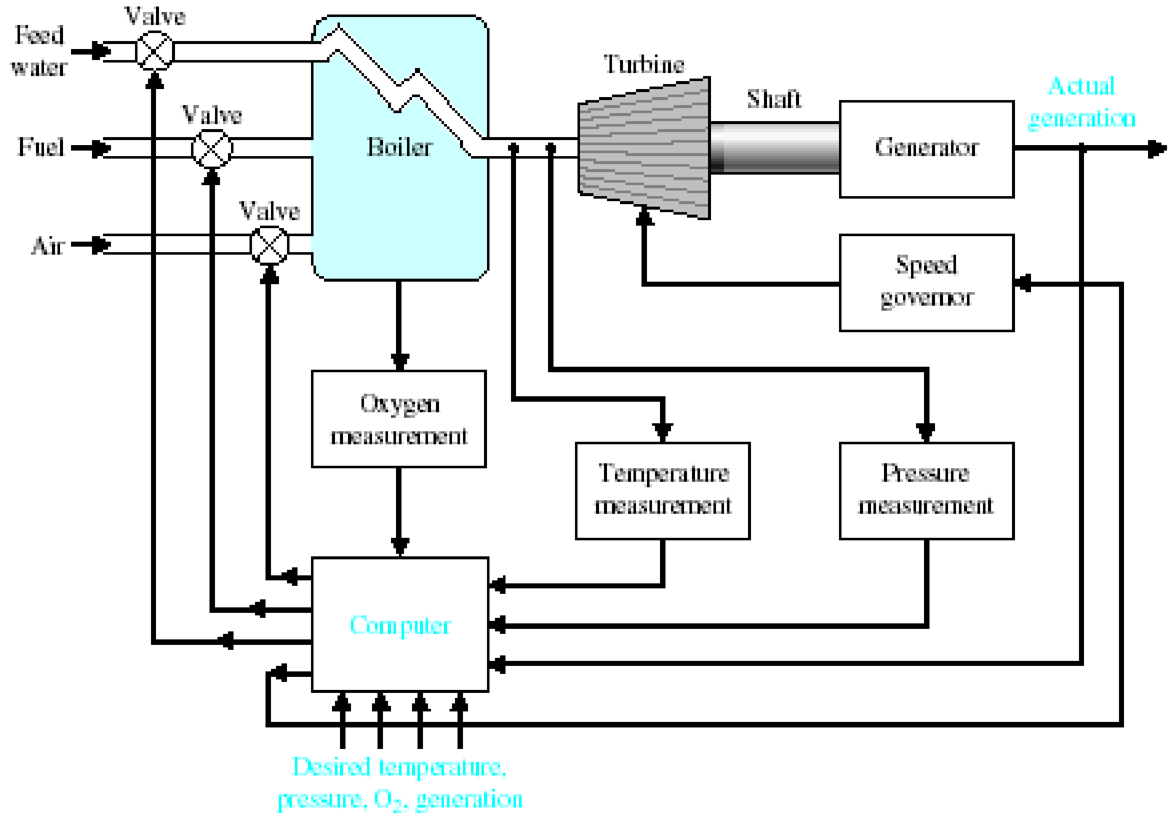
A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.

Examples of Modern Control Systems



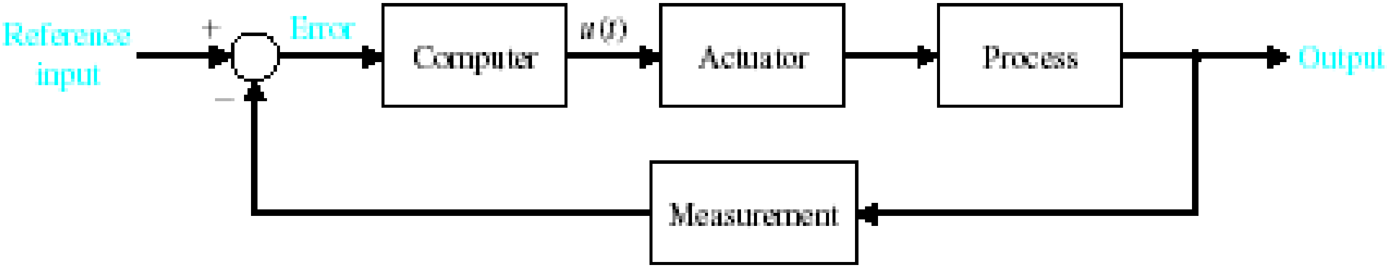
A three-axis control system for inspecting individual semiconductor wafers with a highly sensitive camera.

Examples of Modern Control Systems



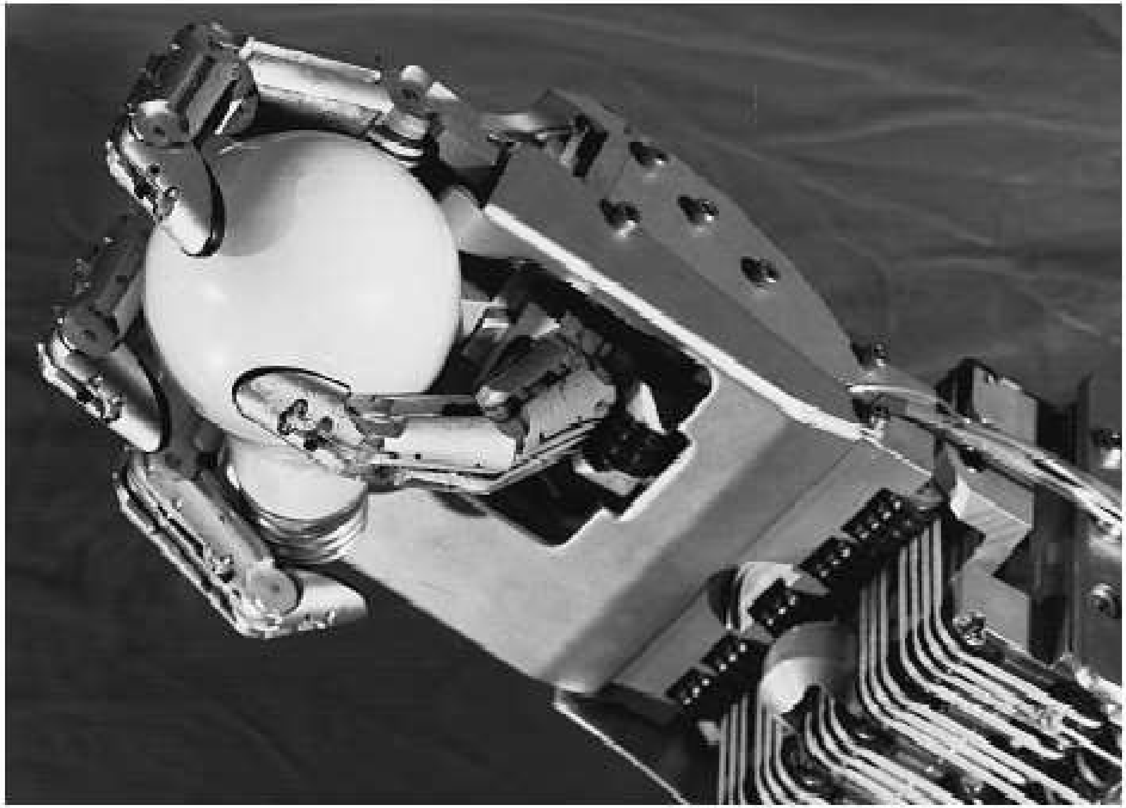
Coordinated control system for a boiler-generator.

Examples of Modern Control Systems



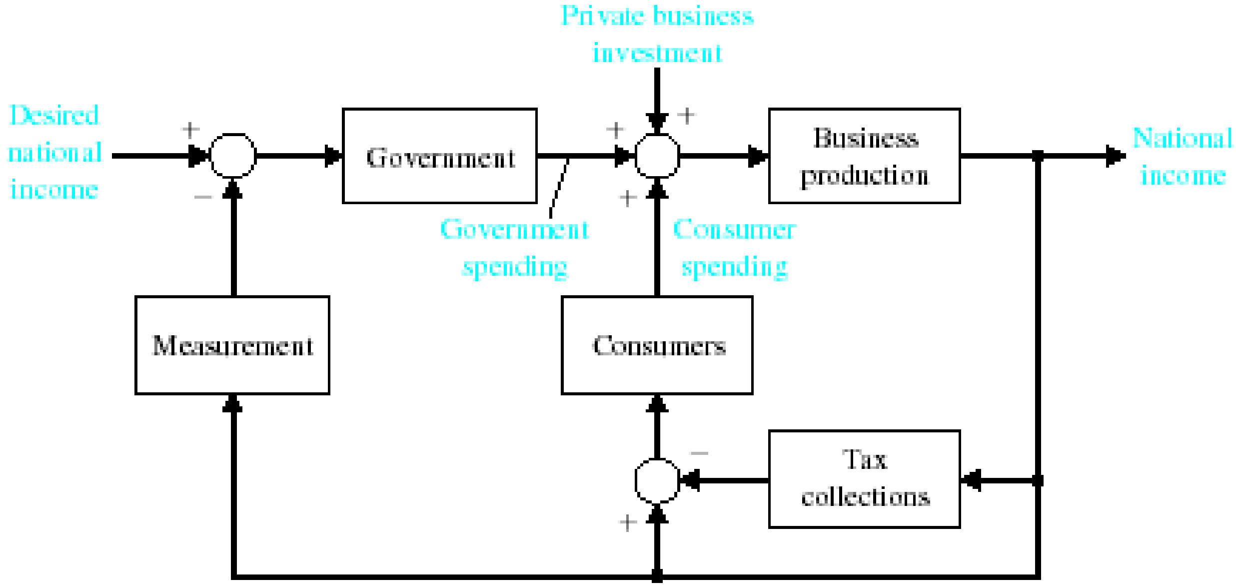
A computer control system.

Examples of Modern Control Systems



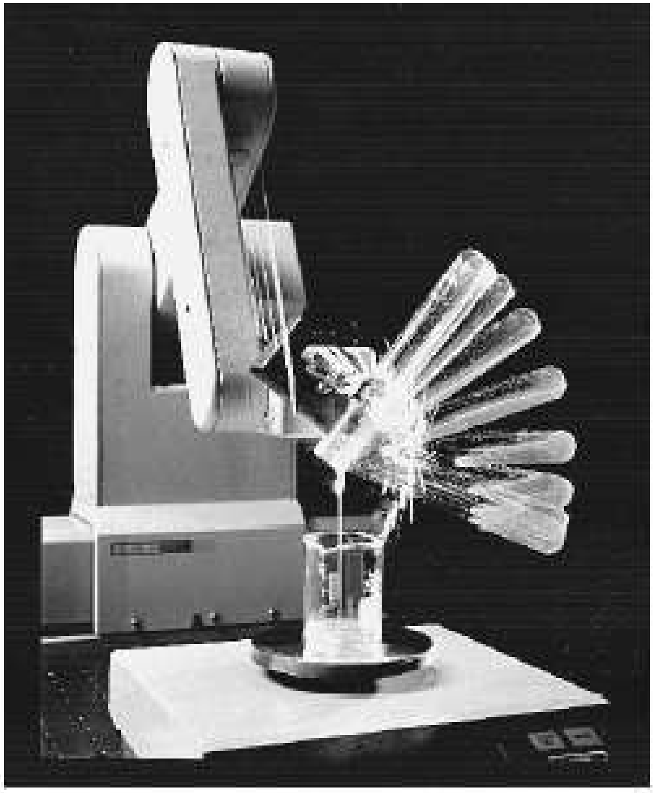
The Utah/MIT Dextrous Robotic Hand: A dextrous robotic hand having 18 degrees of freedom, developed as a research tool by the Center for Engineering Design at the University of Utah and the Artificial Intelligence Laboratory at MIT. It is controlled by five Motorola 68000 microprocessors and actuated by 36 high-performance electropneumatic actuators via high-strength polymeric tendons. The hand has three fingers and a thumb. It uses touch sensors and tendons for control.
(Photograph by Michael Milochik. Courtesy of University of Utah.)

Examples of Modern Control Systems



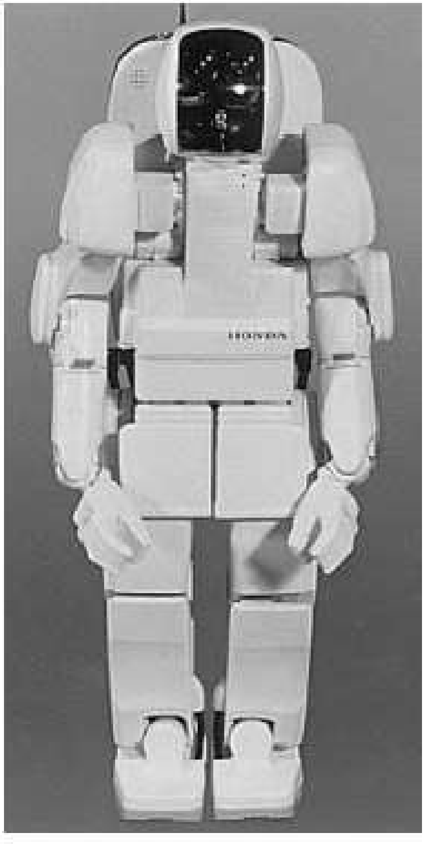
A feedback control system model of the national income.

Examples of Modern Control Systems



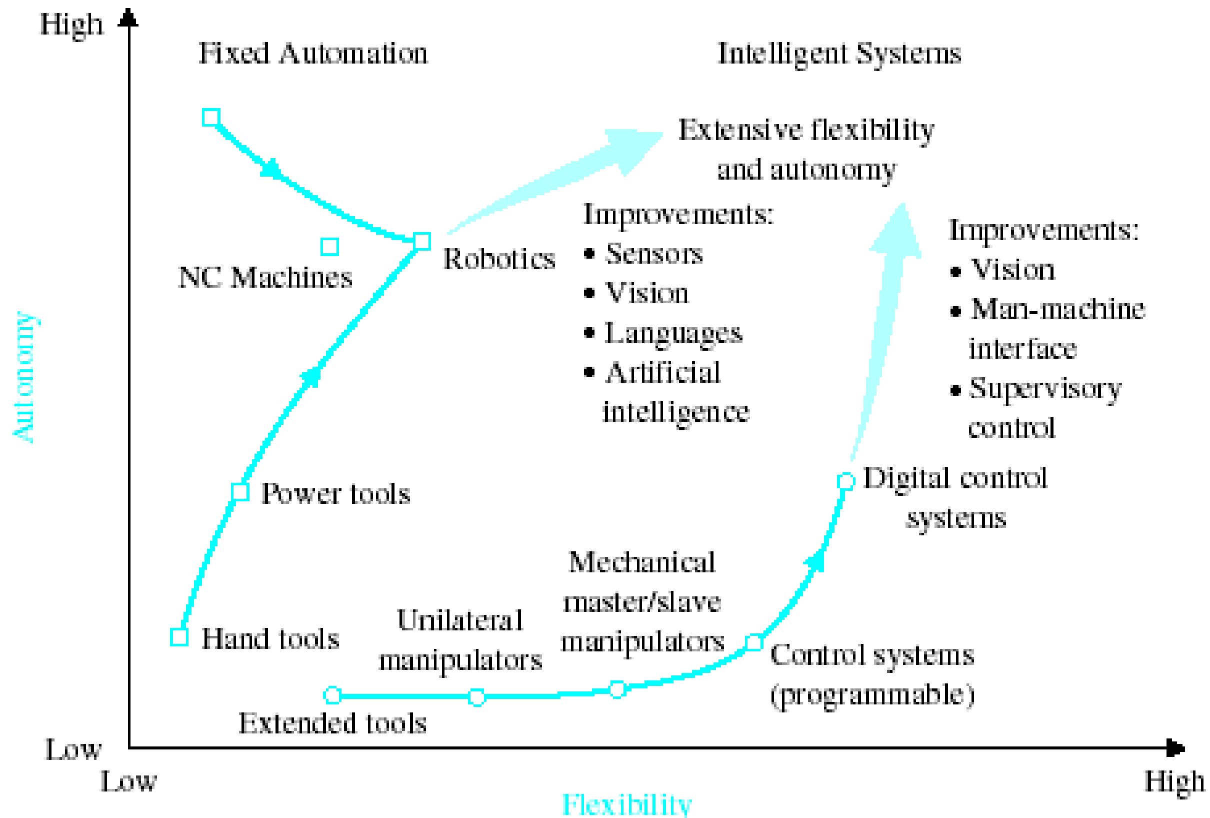
A laboratory robot used for sample preparation. The robot manipulates small objects, such as test tubes, and probes in and out of tight places at relatively high speeds [41]. (© Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)

The Future of Control Systems



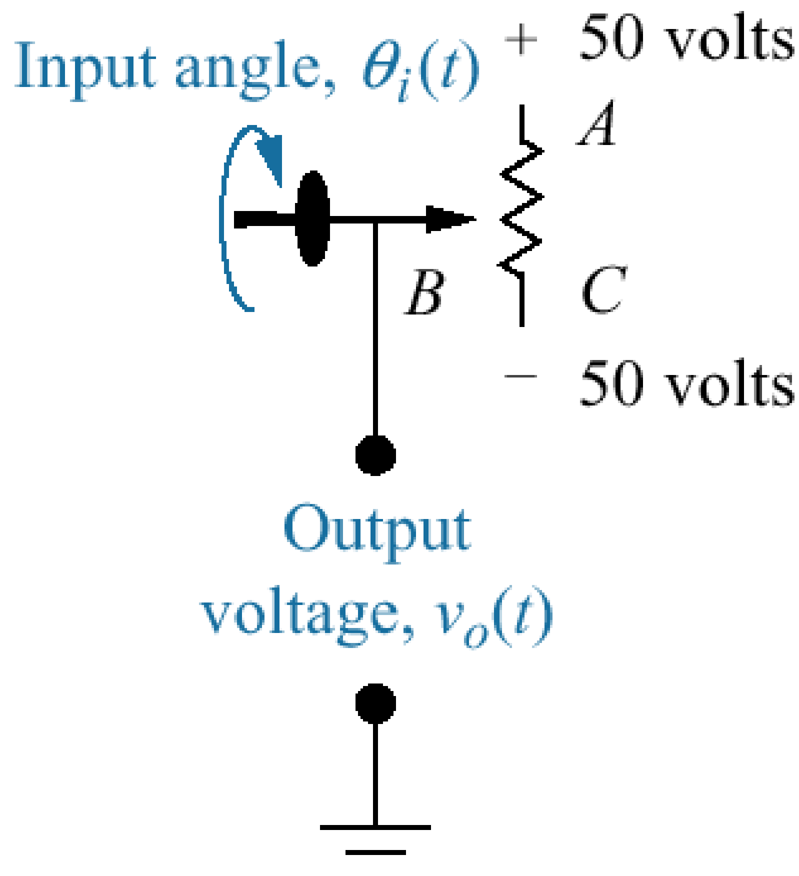
The Honda P3 humanoid robot. P3 walks, climbs stairs and turns corners.
Photo courtesy of American Honda Motor, Inc.

The Future of Control Systems

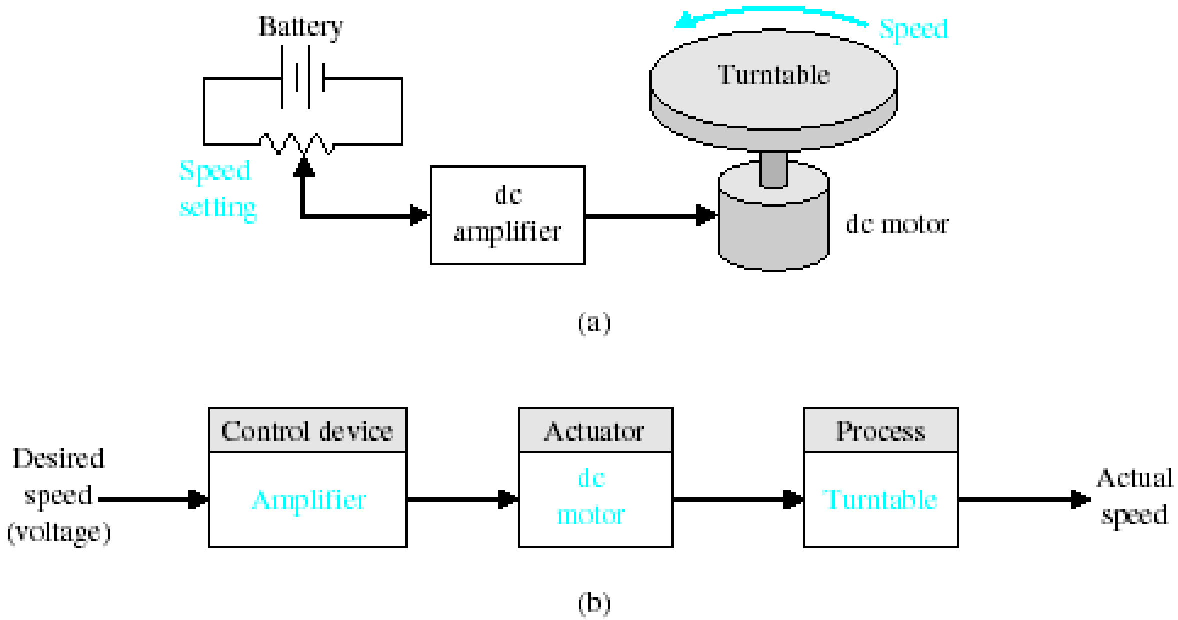


Future evolution of control systems and robotics.

Design Example

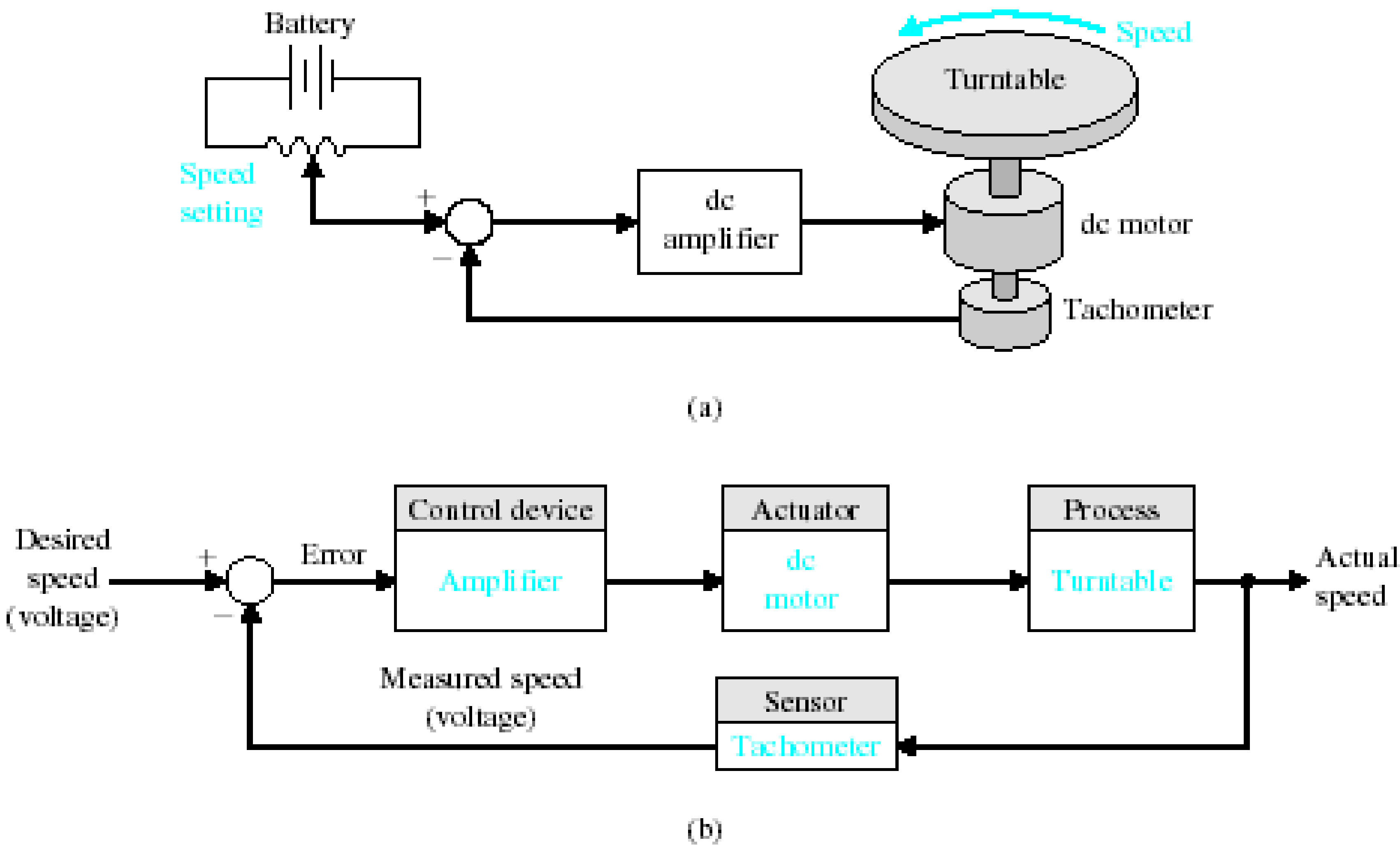


Design Example



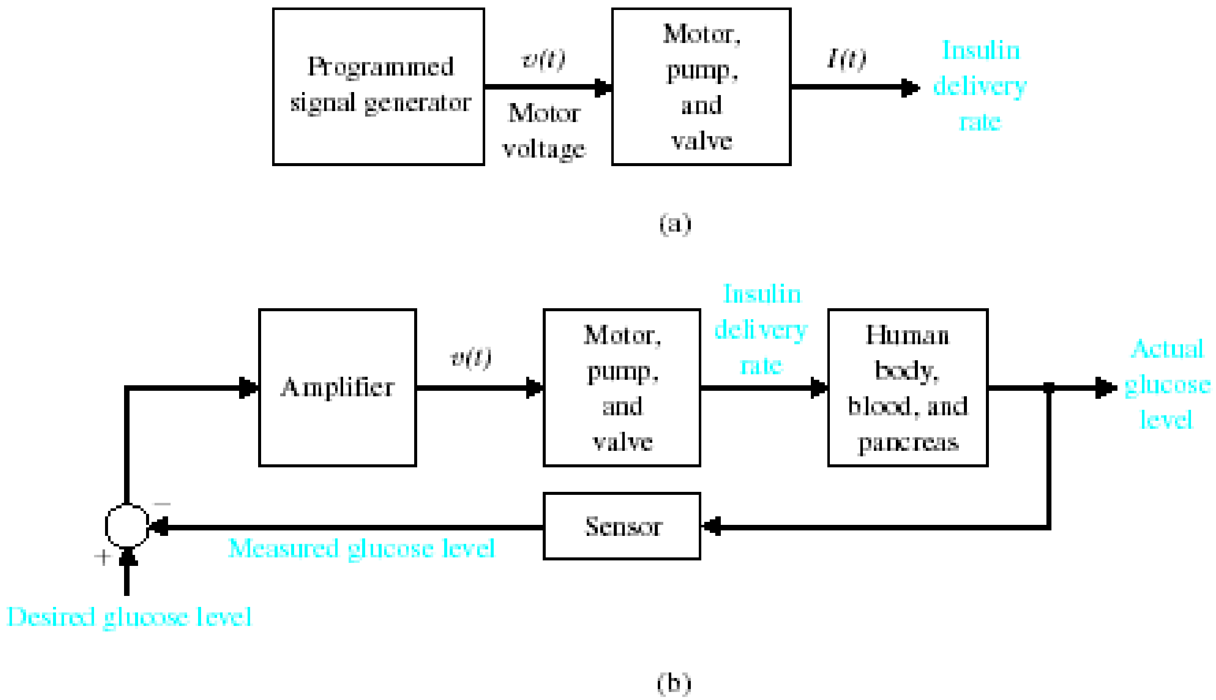
(a) Open-loop (without feedback) control of the speed of a turntable.
(b) Block diagram model.

Design Example



(a) Closed-loop control of the speed of a turntable.
(b) Block diagram model.

Design Example



(a) Open-loop (without feedback) control and
(b) closed-loop control of blood glucose.

The Stability of Linear Feedback Systems

The issue of ensuring the stability of a closed-loop feedback system is central to control system design. Knowing that an unstable closed-loop system is generally of no practical value, we seek methods to help us analyze and design stable systems. A stable system should exhibit a bounded output if the corresponding input is bounded. This is known as bounded-input, bounded-output stability and is one of the main topics of this chapter.

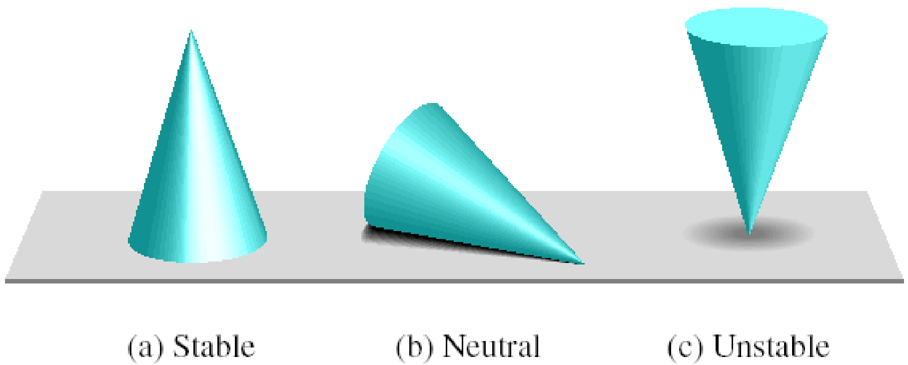
The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function. The Routh–Hurwitz method is introduced as a useful tool for assessing system stability. The technique allows us to compute the number of roots of the characteristic equation in the right half-plane without actually computing the values of the roots. Thus we can determine stability without the added computational burden of determining characteristic root locations. This gives us a design method for determining values of certain system parameters that will lead to closed-loop stability. For stable systems we will introduce the notion of relative stability, which allows us to characterize the degree of stability.

The Concept of Stability

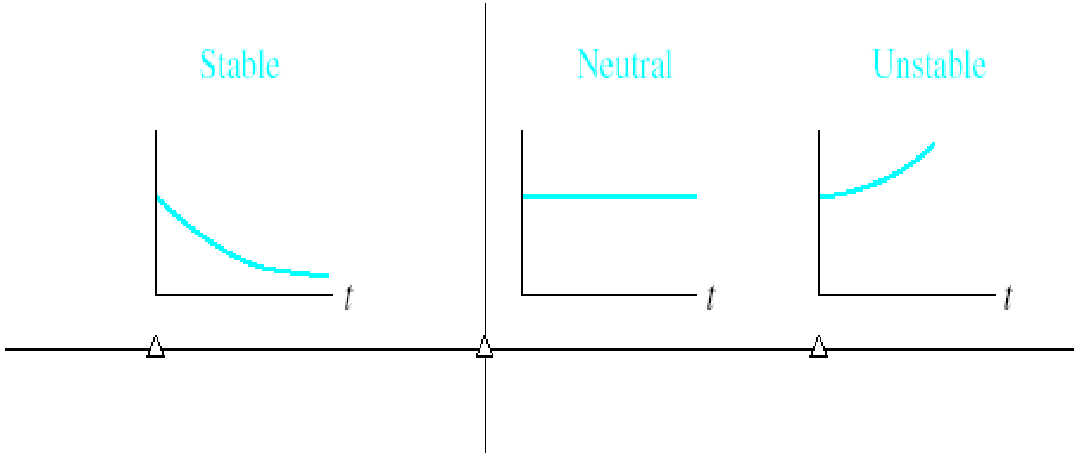
A stable system is a dynamic system with a bounded response to a bounded input.

Absolute stability is a stable/not stable characterization for a closed-loop feedback system. Given that a system is stable we can further characterize the degree of stability, or the relative stability.

The Concept of Stability



The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

The Routh-Hurwitz Stability Criterion

It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are in the left-hand plane.

These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system.

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

The Routh-Hurwitz Stability Criterion

Characteristic equation, q(s)

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

Routh array

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
s^0 & h_{n-1} & &
\end{array}$$

The Routh-Hurwitz criterion states that the number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

The Routh-Hurwitz Stability Criterion

Case One: No element in the first column is zero.

Example 6.1 Second-order system

The Characteristic polynomial of a second-order system is:

$$q(s) = a_2 \cdot s^2 + a_1 \cdot s + a_0$$

The Routh array is written as:

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

where:

$$b_1 = \frac{a_1 \cdot a_0 - (0) \cdot a_2}{a_1} = a_0$$

Therefore the requirement for a stable second-order system is simply that all coefficients be positive or all the coefficients be negative.

The Routh-Hurwitz Stability Criterion

Case Two: Zeros in the first column while some elements of the row containing a zero in the first column are nonzero.

If only one element in the array is zero, it may be replaced with a small positive number ε that is allowed to approach zero after completing the array.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is then:

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & b_1 & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s^1 & d_1 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

where:

$$b_1 = \frac{2\cdot 2 - 1\cdot 4}{2} = 0 = \varepsilon \qquad c_1 = \frac{4\varepsilon - 2\cdot 6}{\varepsilon} = \frac{-12}{\varepsilon} \qquad d_1 = \frac{6\cdot c_1 - 10\varepsilon}{c_1} = 6$$

There are two sign changes in the first column due to the large negative number calculated for c1. Thus, the system is unstable because two roots lie in the right half of the plane.

The Routh-Hurwitz Stability Criterion

Case Three: Zeros in the first column, and the other elements of the row containing the zero are also zero.

This case occurs when the polynomial q(s) has zeros located symetrically about the origin of the s-plane, such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$. This case is solved using the auxiliary polynomial, U(s), which is located in the row above the row containing the zero entry in the Routh array.

$$q(s) = s^3 + 2\cdot s^2 + 4s + K$$

Routh array:

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \frac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For a stable system we require that $0 < s < 8$

For the marginally stable case, K=8, the s^1 row of the Routh array contains all zeros. The auxiliary plynomial comes from the s^2 row.

$$U(s) = 2s^2 + Ks^0 = 2\cdot s^2 + 8 = 2\left(s^2 + 4\right) = 2(s + j\cdot 2)(s - j\cdot 2)$$

It can be proven that U(s) is a factor of the characteristic polynomial:

$$\frac{q(s)}{U(s)} = \frac{s + 2}{2}$$

Thus, when K=8, the factors of the characteristic polynomial are:

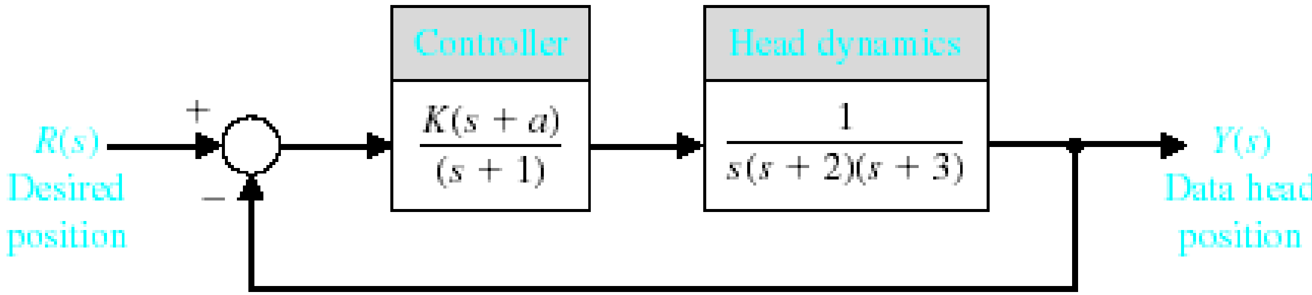
$$q(s) = (s + 2)(s + j\cdot 2)(s - j\cdot 2)$$

The Routh-Hurwitz Stability Criterion

Case Four: Repeated roots of the characteristic equation on the jw-axis.

With simple roots on the jw-axis, the system will have a marginally stable behavior. This is not the case if the roots are repeated. Repeated roots on the jw-axis will cause the system to be unstable. Unfortunately, the routh-array will fail to reveal this instability.

Example



Welding head position control.

Using block diagram reduction we find that: $q(s) = s^4 + 6s^3 + 11s^2 + (K+6)s + Ka$

The Routh array is then:

$$
\begin{array}{c|ccc}
s^4 & 1 & 11 & Ka \\
s^3 & 6 & (K+6) & \\
s^2 & b_3 & Ka & \\
s^1 & c_3 & & \\
s^0 & Ka & &
\end{array}
$$

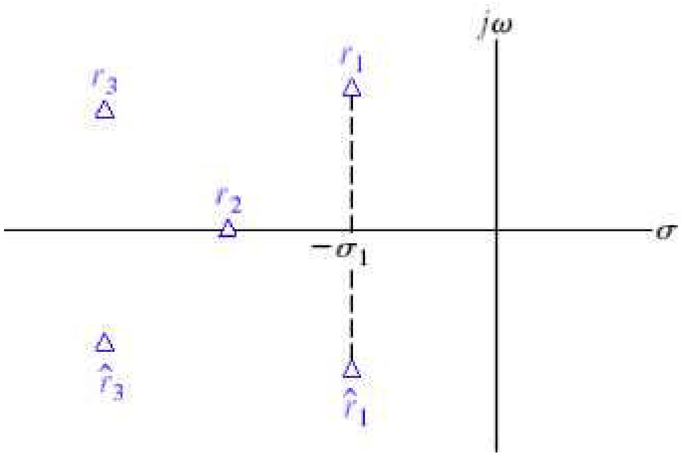
where: $b_3 = \dfrac{60 - K}{6}$ and $c_3 = \dfrac{b_3(K+6) - 6 \cdot Ka}{b_3}$

For the system to be stable both b_3 and c_3 must be positive.

Using these equations a relationship can be determined for K and

The Relative Stability of Feedback Control Systems

It is often necessary to know the relative damping of each root to the characteristic equation. Relative system stability can be measured by observing the relative real part of each root. In this diagram r2 is relatively more stable than the pair of roots labeled r1.



One method of determining the relative stability of each root is to use an axis shift in the s-domain and then use the Routh array as shown in Example 6.6 of the text.

Design Example:

The characteristic equation of this system is:

$$1 + G_c \cdot G(s) = 0$$

or

$$1 + \frac{K(s + a)}{s(s + 1)(s + 2)(s + 5)} = 0$$

Thus,

$$s(s + 1)(s + 2)(s + 5) + K(s + a) = 0$$

or

$$s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0$$

To determine a stable region for the system, we establish the Routh array a

$$\begin{array}{c|ccc}
s^4 & 1 & 17 & Ka \\
s^3 & 8 & (K+10) & 0 \\
s^2 & b_3 & Ka & \\
s^1 & c_3 & & \\
s^0 & Ka & &
\end{array}$$

where

$$b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}$$

Design Example:

$$\begin{array}{c|ccc} s^4 & 1 & 17 & Ka \\ s^3 & 8 & (K+10) & 0 \\ s^2 & b_3 & Ka & \\ s^1 & c_3 & & \\ s^0 & Ka & & \end{array}$$

where

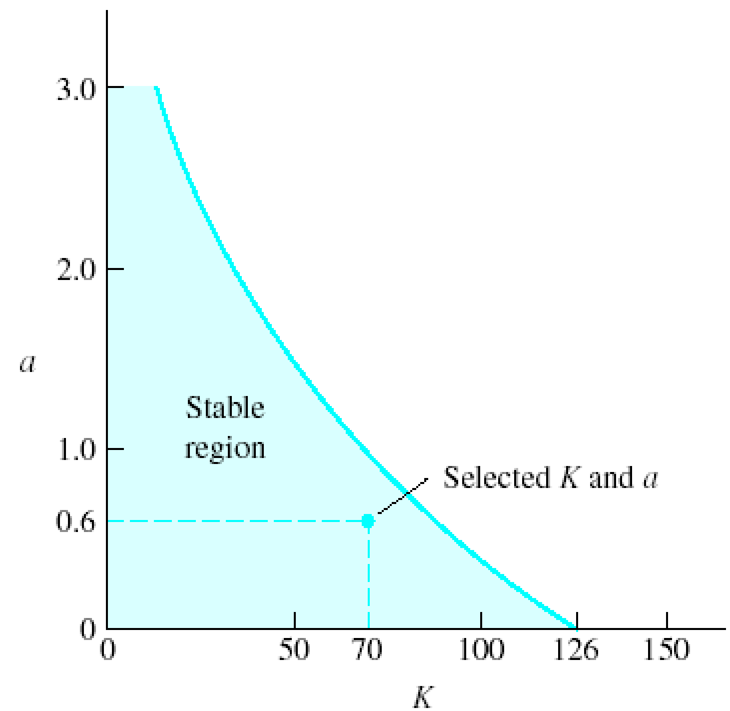
$$b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}$$

Therefore,

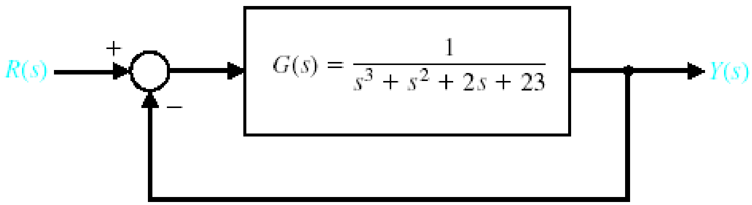
$$K < 126$$

$$K \cdot a > 0$$

$$(K + 10)(126 - K) - 64Ka > 0$$



System Stability Using MATLAB



Closed-loop control system with $T(s) = Y(s)/R(s) = 1/(s^3 + s^2 + 2s$

s^3	1	2
s^2	1	24
s^1	−22	0
s^0	24	0

1st sign change

2nd sign change

```
>>numg=[1]; deng=[1 1 2 23]; sysg=tf(numg,deng);
>>sys=feedback(sysg,[1]);
>>pole(sys)


ans =


  -3.0000
   1.0000 + 2.6458i
   1.0000 - 2.6458i
```

Unstable poles

Root Locus Techniques

Objectives

- Definition of a root locus
- How to sketch a root locus
- How to refine your sketch of a root locus
- How to use the root locus to find the poles of a closed-loop system
- How to use the root locus to describe qualitatively the changes in transient response and stability of a system as a system parameter is varied
- How to use the root locus to design a parameter value to meet a transient response specification for systems of order 2 and higher

Introduction

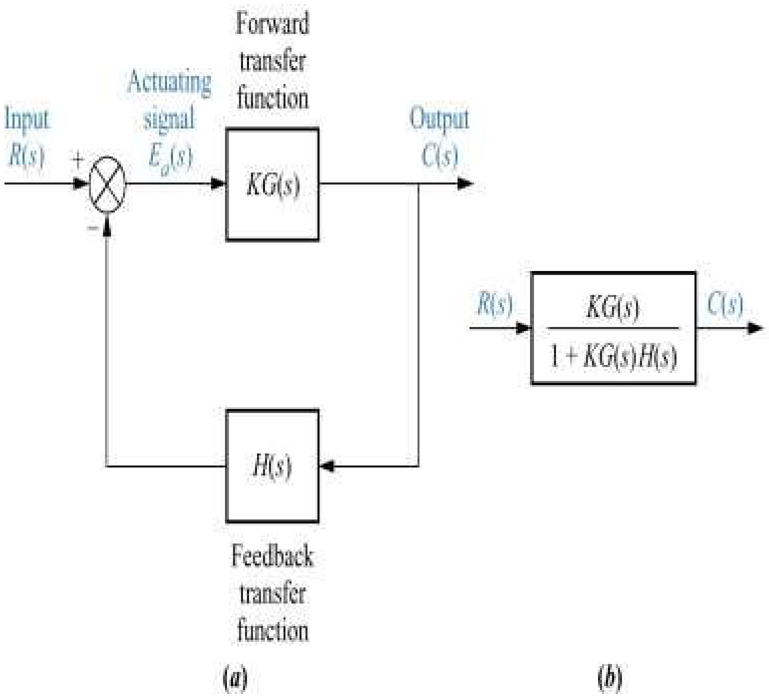
- What is root locus?
  - Root locus is a graphical representation of the closed-loop poles as a system parameter is varied
  - It can be used to describe qualitatively the performance of a system as various parameters are changed
  - It gives graphic representation of a system's transient response and also stability
  - We can see the range of stability, instability, and the conditions that cause a system to break into oscillation

The Control System Problem

- The poles of the open loop transfer function are easily found by inspection and they do not change with changes in system gain. But the poles of the closed loop transfer function are more difficult to find and they change with changes in system gain

- Consider the closed loop system in the next figure
  - a) Closed loop system
  - b) Equivalent transfer function

Note: KG(s)H(s) = Open Loop Transfer Function, or loop gain



If

$$G(s) = N_G(s) / D_G(s)$$

And

$$H(s) = N_H(s) / D_H(s)$$

Then →

$$T(s) = KG(s) / 1 + K(s)H(s)$$

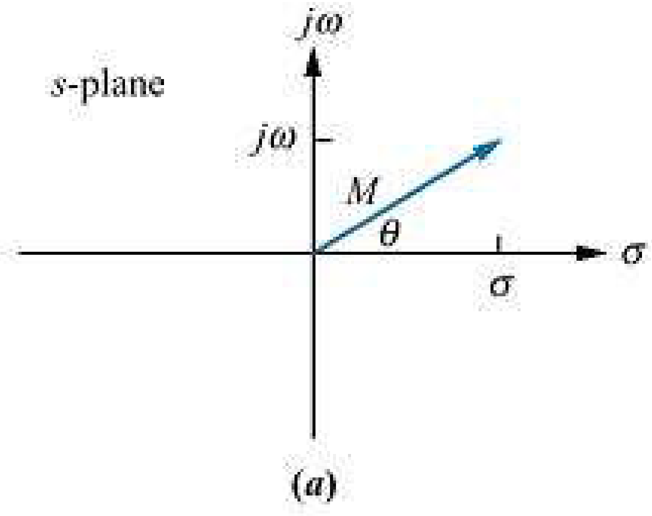
Therefore →

$$\mathbf{T(s) = KN_G(s)D_H(s) / D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

- Observations:
  - The zero of T(s) consist of the zeros of G(s) and The poles of H(s)
  - The poles of T(s) are not immediately known without factoring the denominator and they are a function of K
  - Since the system's performance depends on the knowledge of the poles' location, we will not be able to know the system performance readily
  - Root locus can be used to give us a picture of the poles of T(s) as the system gain, K, Varies

Vector representation of complex number

- Vector has a magnitude and a direction
- Complex number (σ + jω) can be described in Cartesian coordinates or in polar form. It can also be represented by a vector
- If a complex number is substituted into a complex function, F(s), another complex number will result



Example:

If

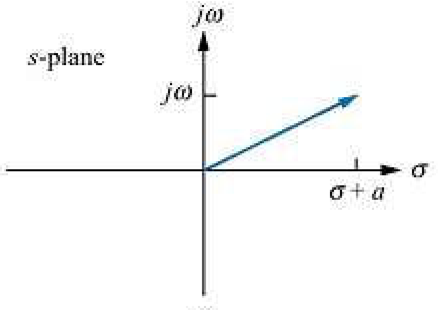
$s = (\sigma + j\omega)$ is substituted into $F(s) = (s + a)$

Then

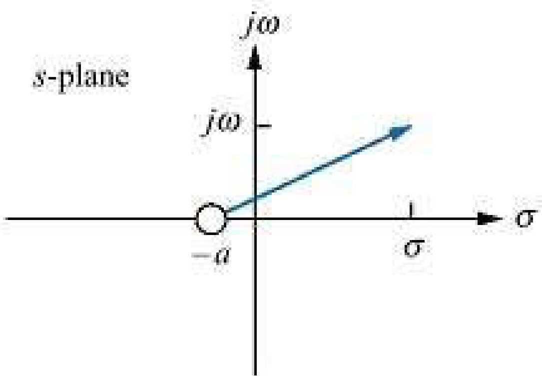
$$F(s) = (\sigma + a) + j\omega$$

Therefore

$(s + a)$ is a complex number and can be represented by a vector drawn from the zero of the function to point S



(b)

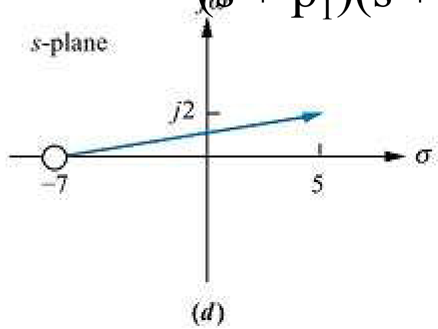


(c)

Draw the vector representation of $(s + 7)|_{s \to 5 + j2}$

Recall

$$F(s) = \frac{(s + z_1)(s + z_2)\ldots}{(s + p_1)(s + p_2)\ldots}$$



Or

$$F(s) = \prod_{i=1}^{m} (s + z_i) \Big/ \prod_{j=1}^{n} (s + p_j)$$

Where

$\prod$ = Product

Or

F(s) = M∟θ (in polar form)

Summary

M = ∏ zero lengths / ∏ pole lengths

$$M = \frac{\prod_{i}^{m} |(s + z_i)|}{\prod_{j}^{n} |(s + p_j)|}$$

Θ = Σ zero angles – Σ pole angles

$$\theta = \sum_{i}^{m} \angle(s + z_i) - \sum_{j}^{n} \angle(s + p_j)$$

Given $F(s) = \dfrac{(s+1)}{s(s+2)}$; Find F(s) at the point s = -3 + j4

Graphically:

For (s +1):

$(s+1)|_{s \to -3+j4}$

$= (-3 + j4) + 1$

$= -2 + j4$

$= 4.47 \angle 116.56^{o}$

Similarly:

$s|_{s \to -3+j4} = 5 \angle 126.9^{o}$

$(s+2)|_{s \to -3+j4} = -1 + j4$

$= 4.12 \angle 104.03^{o}$

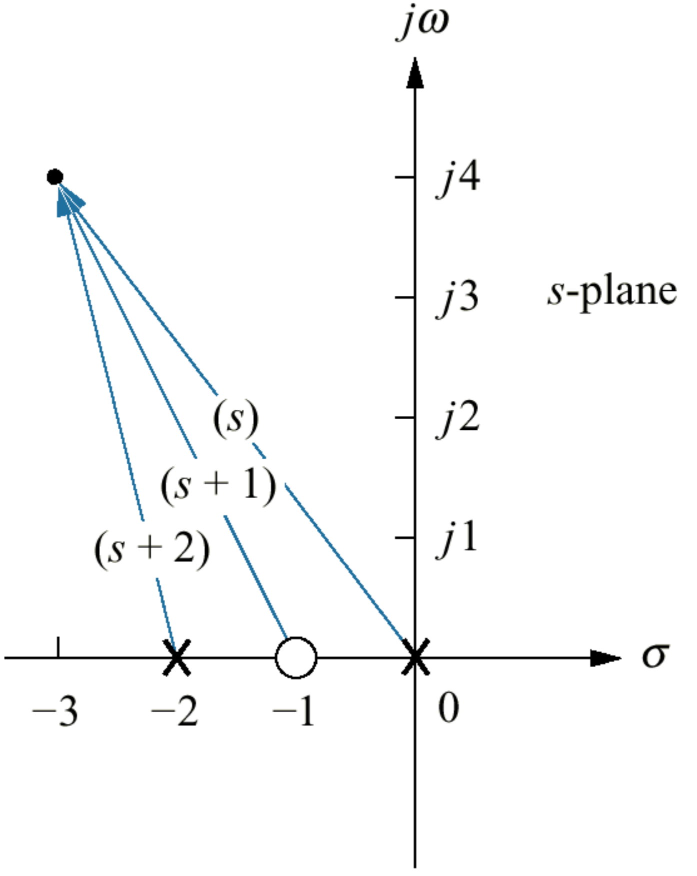
Therefore

$M \angle \theta = F(s)|_{s \to -3+j4}$

$= 4.47 / 5(4.12)$

$\angle 116.56^{o} - 126.9^{o} + 104.03^{o}$

$= 0.217 \angle -114.3^{o}$



Defining the Root Locus

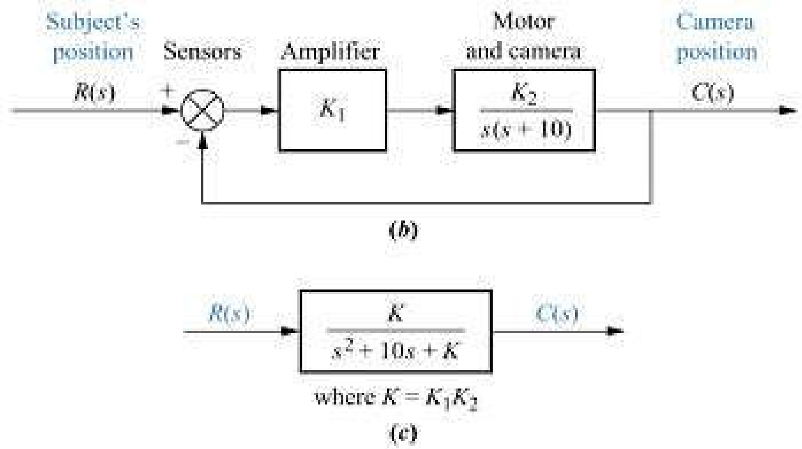
Consider the system represented by block diagram next:

The C.L.T.F.

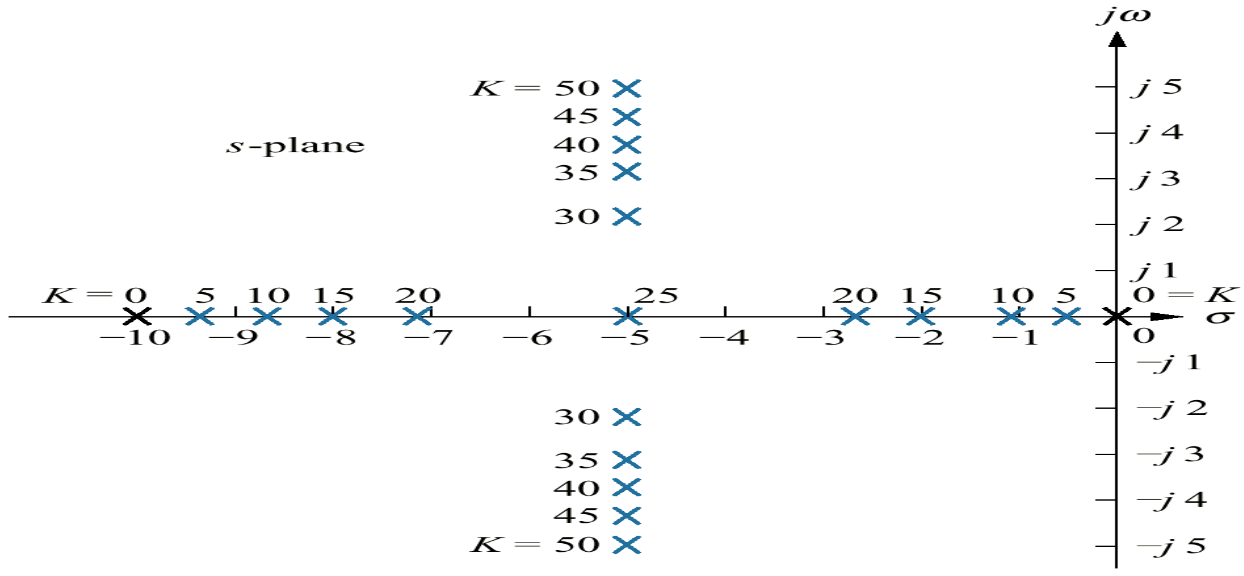
= K / s^2 +10s + K

Where K = K1K2

If we plot the poles of the C.L.T.F. for value K = 0 → 50, we will obtain the following plots

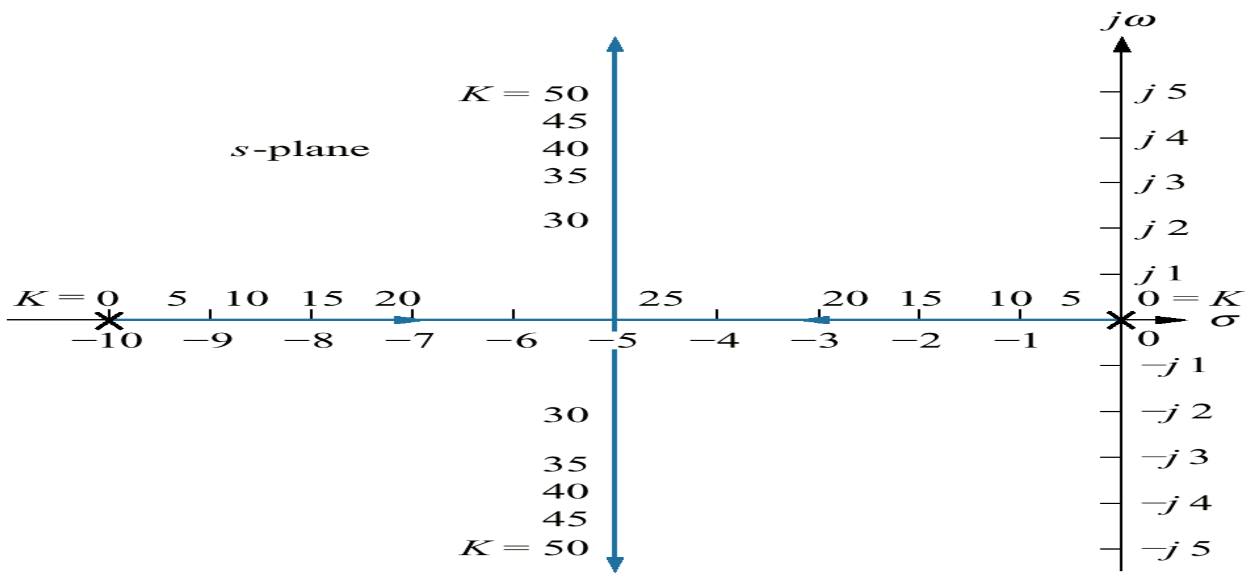


K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



jω
j 5
j 4
j 3
j 2
j 1
s-plane
K = 50
45
40
35
30
K = 0
5
10
15
20
25
20
15
10
5
0 = K
σ
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
−j 1
−j 2
−j 3
−j 4
−j 5

(a)



jω
j 5
j 4
j 3
j 2
j 1
s-plane
K = 50
45
40
35
30
K = 0
5
10
15
20
25
20
15
10
5
0 = K
σ
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
−j 1
−j 2
−j 3
−j 4
−j 5

(b)

Observations:

- Root locus is the representation of the path of the closed loop poles as the gain is varied
- Root locus show the changes in the transient response as the gain K, varies

For $0 \leq K < 25$

- poles are real and distinct ($j\omega = 0$)
- Overdamped response

For $K = 25$

- Poles are real and multiples
- Critical response

For $25 < K < 50$ (or $K > 25$)

- Poles are complex conjugate
- Underdamped response
- Since T_s is inversely ∞ to the real part of the pole and the real part remains the same for $K > 25$
- Therefore, the settling time, T_s, remains the same regardless of the value of gain (Note that $T_s = 4 / \sigma_d$)

For $K > 25$

- as the gain increases, the damping ratio, $\zeta = \cos\theta$ decreases and thus the %OS decreases too
- Note: %OS = e-[ζπ/√(1-ζ2)]*100
- As the gain increases, the damped freq. of oscillation, ω_d, which is the imaginary part of the complex pole also increase
- Since peak time, $T_p = \pi / \omega_d$, thus an increase in ω_d will result in an increase in Tp
- Finally, since the root locus never crosses over into the RHP, the system is always stable, regardless of the value of gain

Properties of the Root Locus

- Knowing the properties of Root Locus will enable us to sketch the root locus without having to factor the denominator of the closed loop transfer function
- Consider the general representation of the C.L.T.F:

$$T(s) = KG(s) / 1 + KG(s)H(s)$$

A pole exists when the characteristic polynomial in the denominator becomes zero

Hence, $KG(s)H(s) = -1 = 1\angle(2k + 1)180^0$, $k = 0, \pm 1, \pm 2, \ldots$

Or $|KG(s)H(s)| = 1$ and $\angle KG(s)H(s) = (2k + 1)180^0$

Similarly,

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod pole\,length}{\prod zero\,length}$$

Hence, Given the poles & zeros of the open loop Transfer function, KG(s)H(s), a point in the s-plane is on the root locus for a particular value of gain, K, if the angles of the zeros minus the angles of the poles, all drawn to the selected point on the S-plane, add up to $(2k + 1)180^0$

Given a unity feedback system that has a the following forward transfer function:

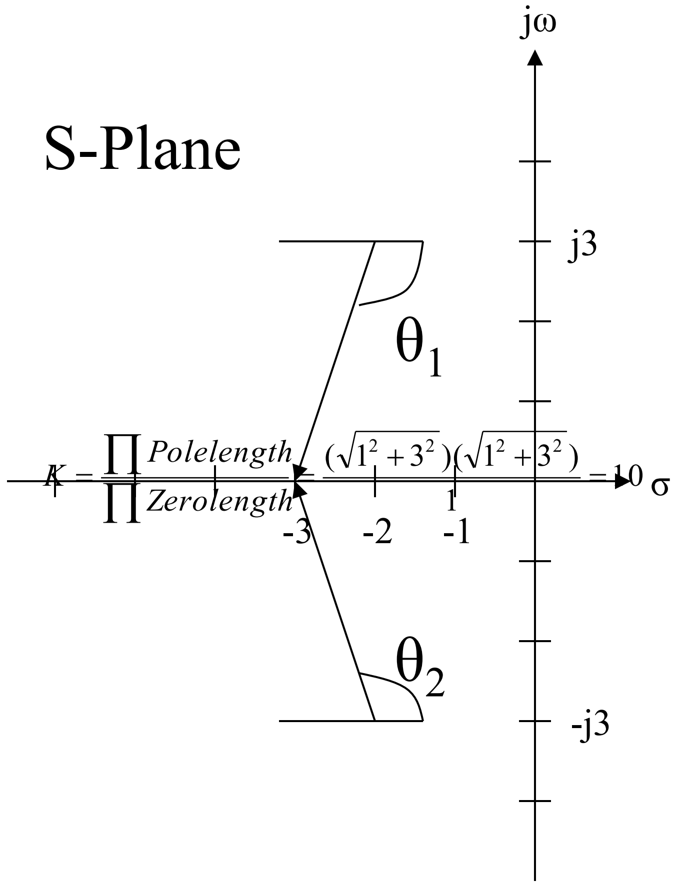
$$G(s) = \frac{K(s+2)}{(S^2+4s+13)}$$

a. Calculate the angle of G(s) at the point (-3 + j0) by finding the algebraic sum of angle of the vectors drawn from the zeros & poles of G(s) to the given point

b. Determine if the point specified in (a) is on the root locus

c. If the point is on the Root Locus, find the gain K using the lengths of the vectors

$$G(s) = \frac{K(s+2)}{(S^2 + 4s + 13)} = \frac{K(s+2)}{(s+2+j3)(s+2-j3)}$$



Σ angles = $180^0 + \theta_1 + \theta_2 = 180^0 - 108.43^0 + 108.43^0 = 180^0$

Or

$\angle G(s)|s=-3j0$

$= \Sigma\theta_{zeros} - \Sigma\theta_{poles}$

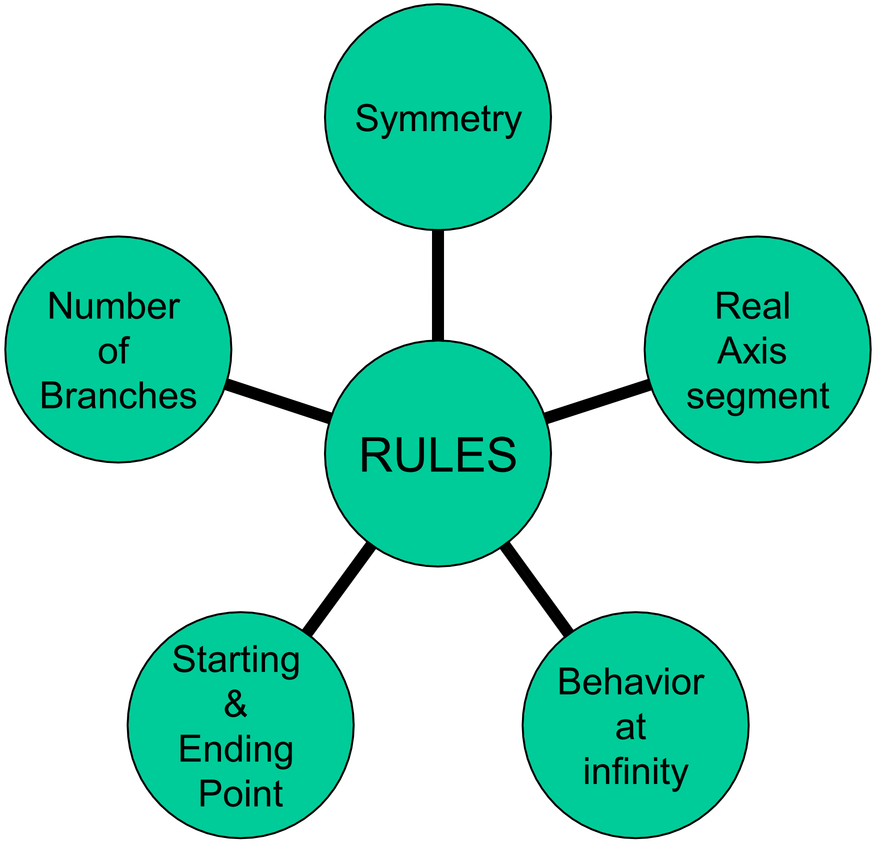
$= 180^0 - (-108.43^0 + 108.43^0)$

$= 180^0$

Since the angle is 180^0, the point is on Root Locus

Sketching the Root Locus

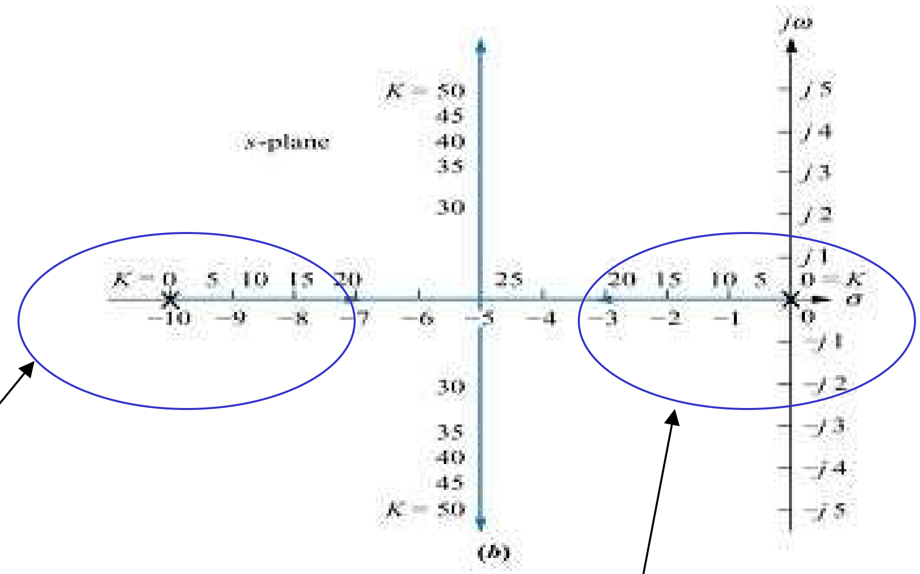
- Based on the properties of root locus, some rules are established to enable us to sketch the Root Locus:



No. of branches

The no. of branches of the R.L equals the number of closed-loop poles. (Since a branch is the path that one poles traverses.)

1st

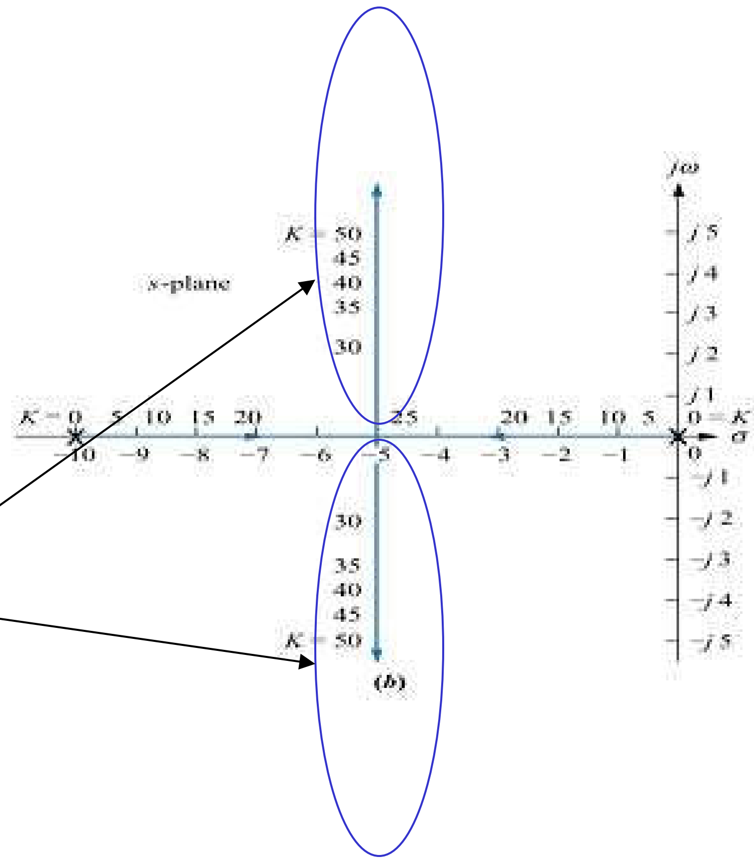


2nd

Symmetry

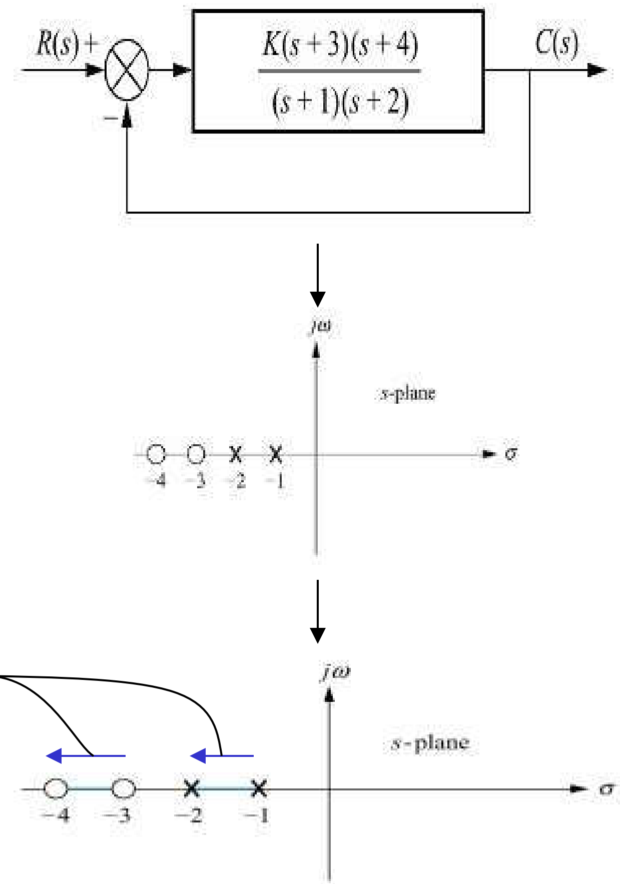
The root locus is symmetrical about the real-axis. (Since complex poles always exist in complex conjugate form.)

Symmetrical about real axis



Real-axis segment

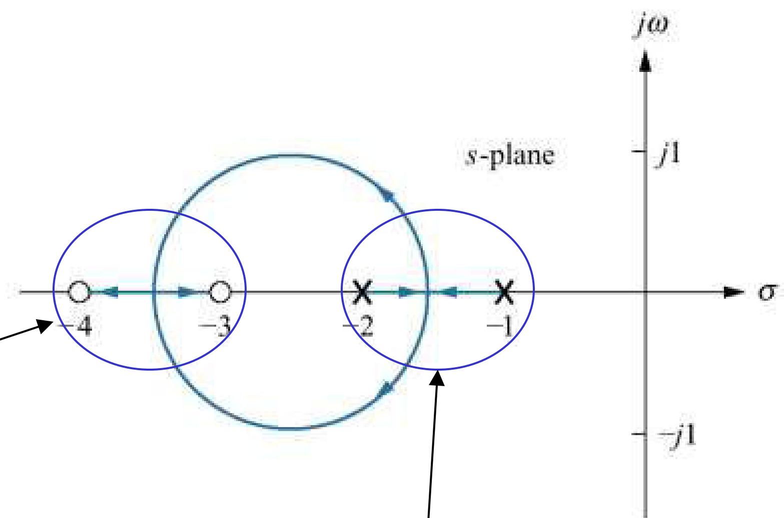
On the real-axis, for K>0, the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros. (Due to the angle property of R-L.)



To the left of an odd number

Starting & Ending Points

The root locus begins at the finite & infinite poles of G(s)H(s) & ends at the finite & infinite zeros of G(s)H(s).



Endin
g

Starti
ng

Concept of Infinite pole & zero

- Infinite pole: If the function approaches ∞ as s approaches ∞, then the function has an infinite pole.
- Infinite zero: If the function approaches zero as s approaches ∞, then, the function has an infinite zero
- Example: $KG(s)H(s) = K / s(s + 1)(s + 2)$
- This function has 3 finite poles at 0, -1, -2 & 3 infinite zeros.
- Every function of s has an equal no. of poles & zeros if we included the infinite poles & zeros as well as the finite poles & zeros.

Behavior at infinity

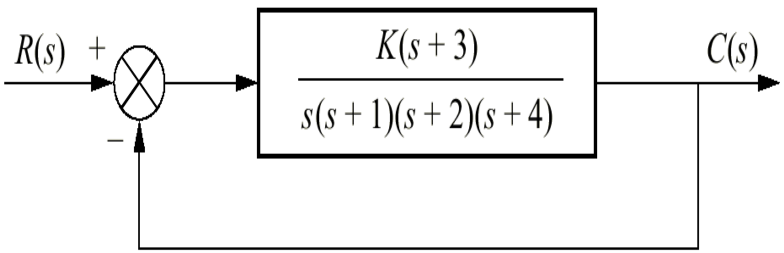
- The root locus approaches straight lines as asymptotes as the locus approaches infinity.
- The equation of the asymptotes is given by the real-axis intercept, σ_a & *angle* θ_a :

$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{\#\ finite\ poles - \#\ finite\ zeros}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\ finite\ poles - \#\ finite\ zeros}$$

- Where $k = 0, \pm 1, \pm 2, \pm\ 3$, and the angle is given in radians w.r.t. the positive extension of the real-axis.

Example: Sketch the root locus for the system shown



$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{\#\ finite\ poles - \#\ finite\ zeros}$$

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\ finite\ poles - \#\ finite\ zeros}$$

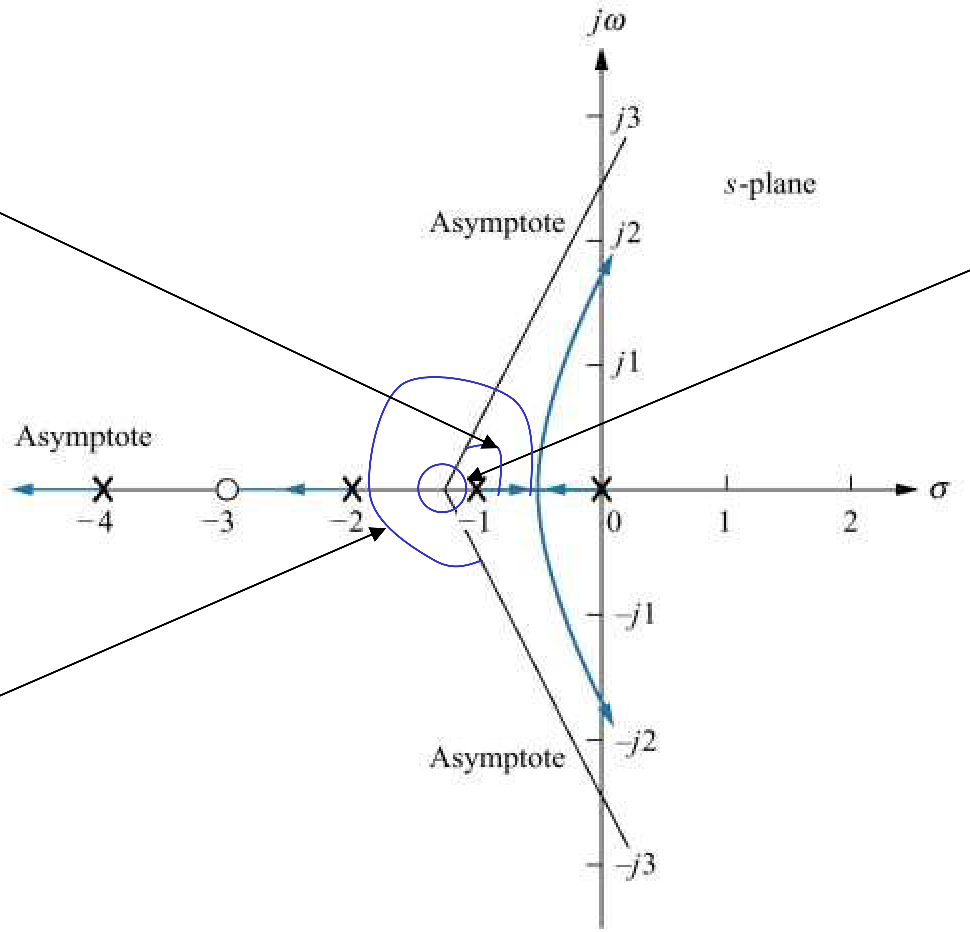
$$= \frac{\pi}{3}\ for\ k = 0$$

$$= \pi\ \ for\ k = 1$$

$$= \frac{5\pi}{3}\ for\ k = 2$$

- Notice that there are 4 finite poles & 1 finite zero.
- Thus there will be 3 infinite zeros.
- Calculate the asymptotes of the infinite zeros:
- Intercept on real-axis.

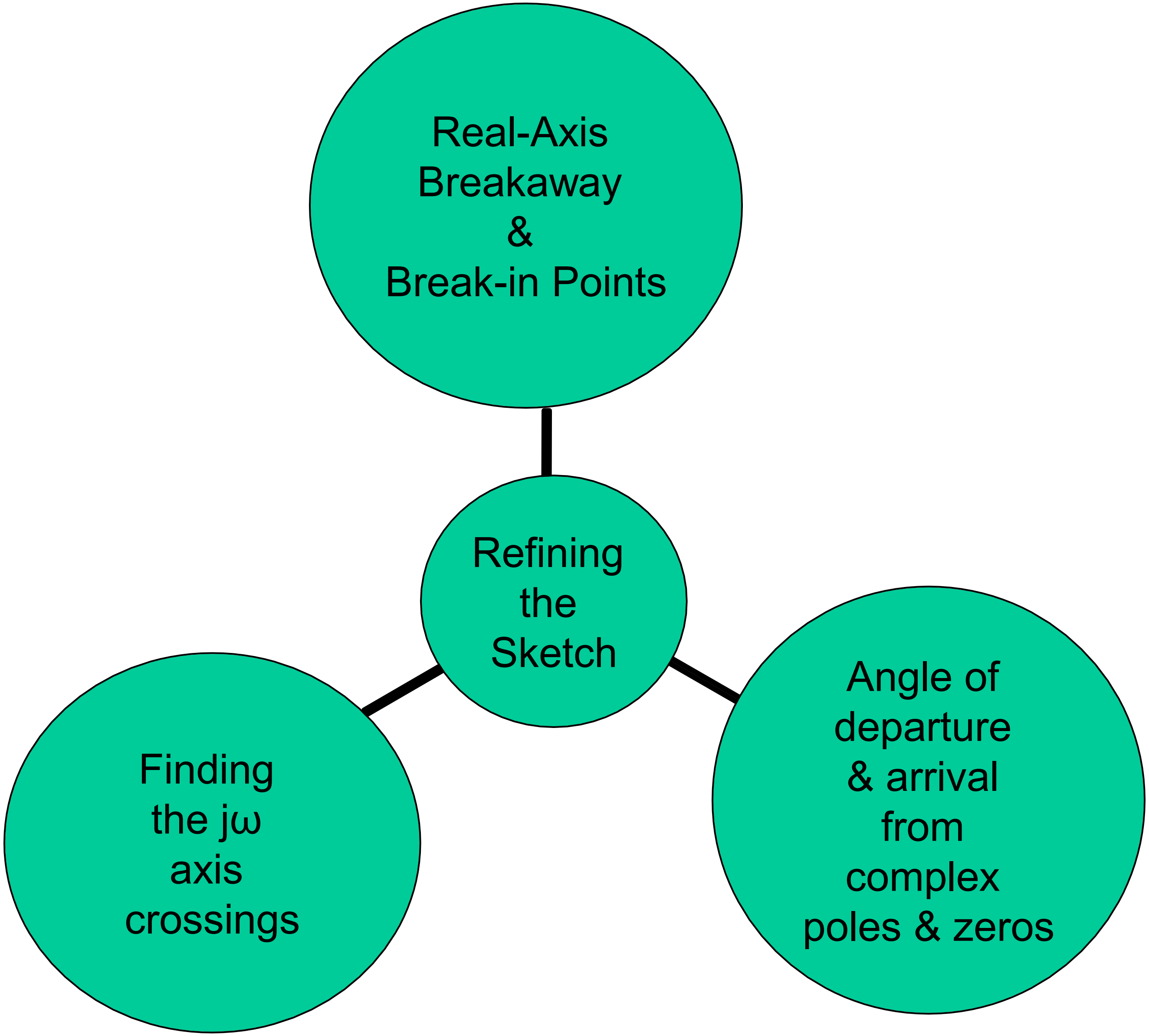
Root locus and asymptotes for the system of previous example



Π /3

Real axis intercept

5Π /3


Real-Axis Breakaway & Break-in Points
Refining the Sketch
Finding the jω axis crossings
Angle of departure & arrival from complex poles & zeros

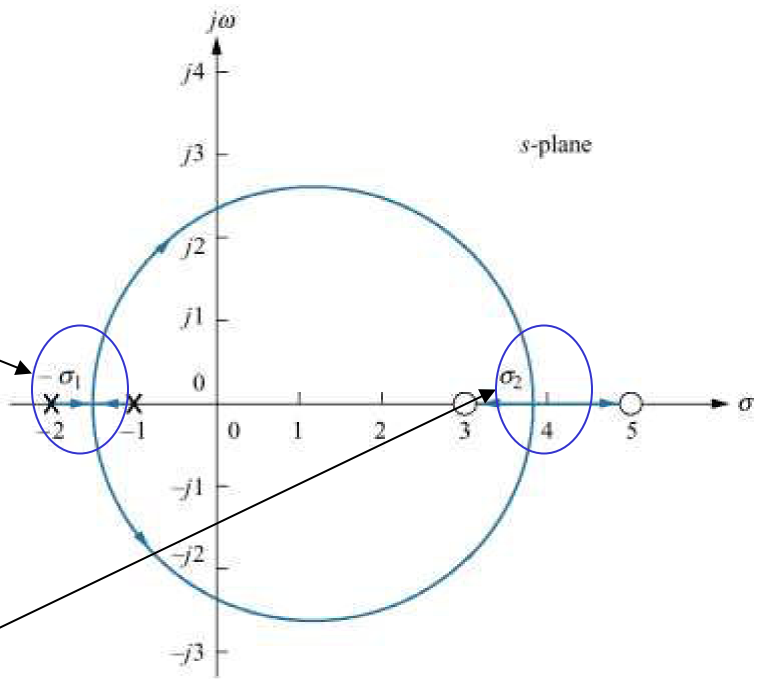
Real-Axis Breakaway & Break-in Points

- Breakaway point is the point where the locus leaves the real-axis.

  (*$-\sigma_1$ in the figure*)

- Break-in point is the point where the locus returns to the real-axis.

  (*σ_2 in the figure*)



Finding the Breakaway & Break-in points via differentiation

- We know that for a point to be on a Root-locus,

  $K = -1 / G(s)H(s)$

- Thus, on the real-axis ($j\omega = 0$),

  $K = -1 / G(\sigma)H(\sigma)$

- Note also that
  - at the breakaway point, K is maximum (for the R-L on the real-axis),
  - and at the break-in point, K is minimum.
- Breakaway & Break-in point can be found by differentiating $K\, G(\sigma)H(\sigma) = -1$ & set it to zero.

Find the breakaway & break-in points for the root locus shown

From the Root Locus

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

$$= \frac{K(s^2 - 8s + 15)}{(s^2 + 3s + 2)}$$

On the root locus & on the real − axis,

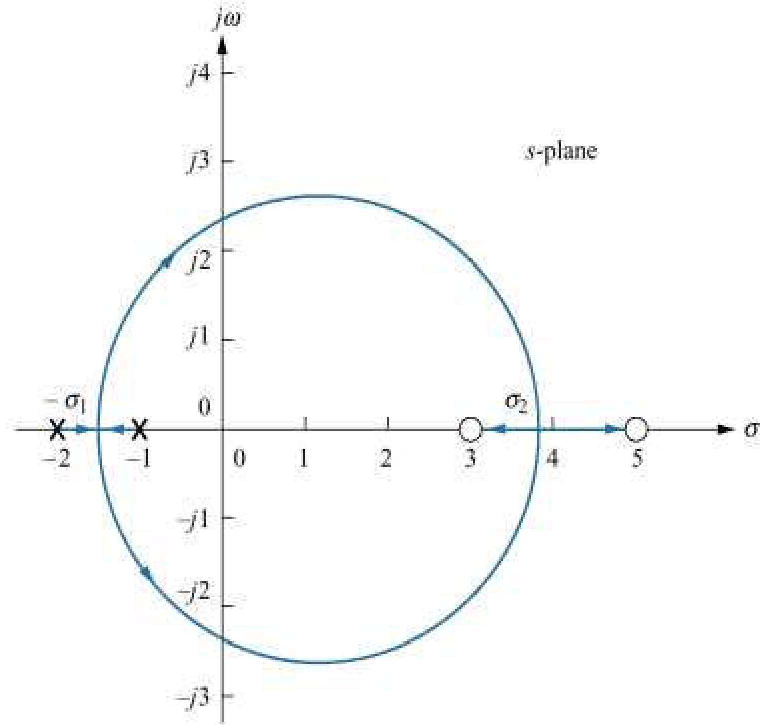
Note: We will get the same result if we take $\frac{\partial G(\sigma)}{d\sigma} = 0$

$$KG(s)H(s) = -1 = KG(\sigma)H(\sigma)$$

$$\frac{K(\sigma^2 - 8\sigma + 15)}{(\sigma^2 + 3\sigma + 2)} = -1$$

$$\frac{\partial k}{d\sigma} = \frac{(11\sigma^2 - 26\sigma + 61)}{(\sigma^2 - 8\sigma + 15)^2} = 0$$

$$\sigma = \underline{-1.45,\ 3.82}$$



Finding Breakaway & Break-in Points by transition method

- This method eliminates the step of differentiation.
- Derivation in Appendix J.2. on CD-Rom.
- This method states that:
  - Breakaway & break-in points satisfy the following relationship:

$$\sum_{1}^{m} \frac{1}{\sigma + z_i} = \sum_{1}^{n} \frac{1}{\sigma + p_i}$$

Where Z_i & P_i are the negative of the zero & pole values, respectively, of G(s)H(s).

Re*peat the previous example with this method*

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2}$$

$$11\sigma^2 + 26\sigma - 61 = 0$$

$$\sigma = \underline{-1.45, 3.82}$$

Finding the jω axis crossings

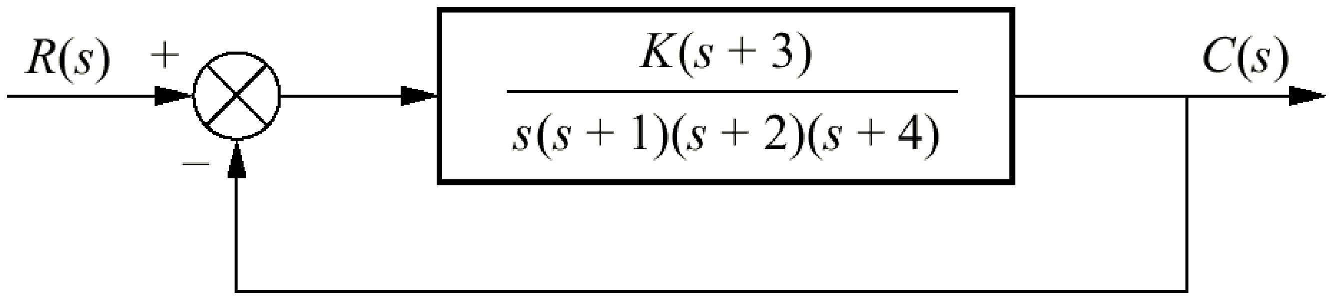
- Jω axis crossing is a point on the R-L that separates the stable operation of the system from the unstable operation.
- The value of ω at the axis crossing yields the frequency of oscillation.
- The gain at the jω axis crossing yields the max. positive gain for system stability.

Jω-axis crossing can be found by using Routh-Hurwitz criterion as follows:

- Forcing a row of zeros in the Routh Table will yield the gain.
- Going back one row to the even polynomial equation & solving for the roots yields the frequency at the imaginary axis crossing.

(Recall that a row of zeros in the Routh Table indicates the existence of poles on the jω axis.)

For the system shown, find the frequency & gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?



$$C.L.T.F \ of \ T(s) = \frac{G(s)}{1+G(s)H(s)}, \ H(s) = 1$$

where

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

Construction of Routh table

s^4	1	14	$3K$
s^3	7	$8 + K$	
s^2	$90 - K$	$21K$	
s^1	$\dfrac{-K^2 - 65K + 720}{90 - K}$		
s^0	$21K$		

Continuation of Previous Problem Solving

For +ve K, only s^1 row can be all zeros.

Let $-K^2 - 65K + 720 / 90 - K = 0$ *to find value of K on jω-axis.*
$-K^2 - 65K + 720 = 0$
$K = 9.65$

To find the frequency on the jω axis crossing, form the even polynomial by using the s^2 row & with K= 9.65,

$(90 - K)s^2 + 21K = 0$
$80.35s^2 + 202.7 = 0$
$s^2 = -202.7 / 80.35$
$s = \pm j1.59$

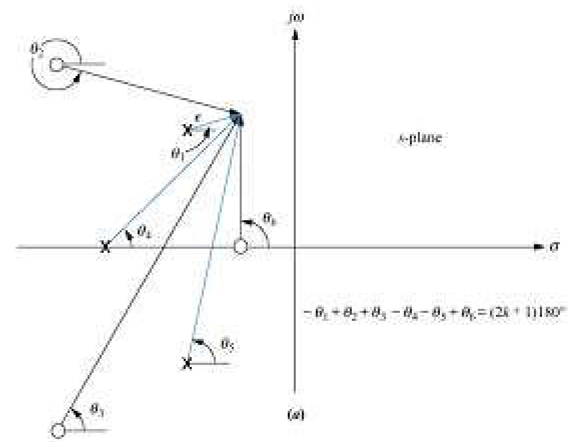
The root-locus crosses the jω axis at $\pm j1.59$ at a gain of 9.65

The system is stable for $0 \le K \le 9.65$

Angle of departure & arrival from complex poles & zeros

- Recall that a condition for a point on the s-plane to be on the root locus is that the angles of the zeros minus the angles of the poles, all drawn to the selected point on the s-plane, add up to

  $(2k + 1)\ 180°$.



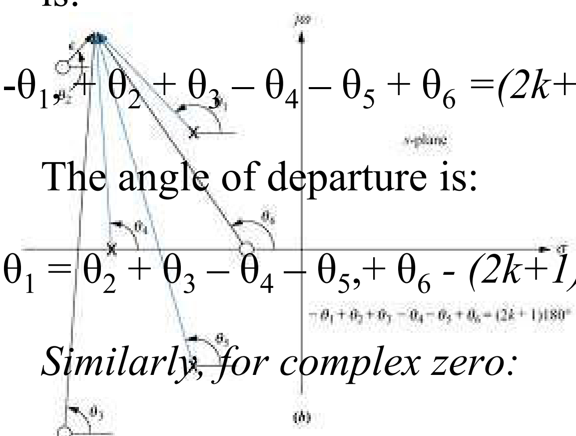
Example

$\angle KG(s)H(s) = (2k + 1)\ 180°$

Consider the next Figure:

Angle of departure & arrival

- Assume ε is a point on the root locus close to a complex pole.
- Sum of all angles drawn from all other poles & zeros to the pole that is near to ε is:



$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+I)180°$

The angle of departure is:

$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5, + \theta_6 - (2k+1)180°$

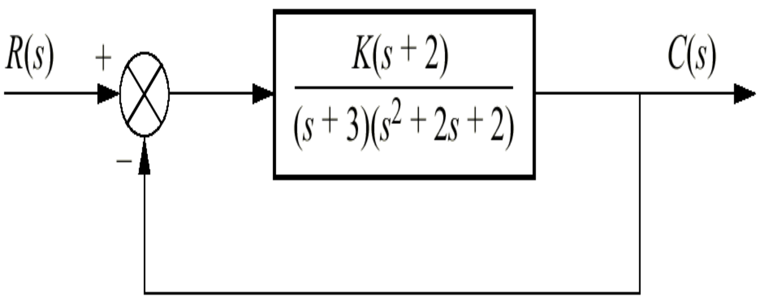
Similarly, for complex zero:

$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+I)180°$

The angle of arrival is:

$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5, - \theta_6 + (2k+1)180°$

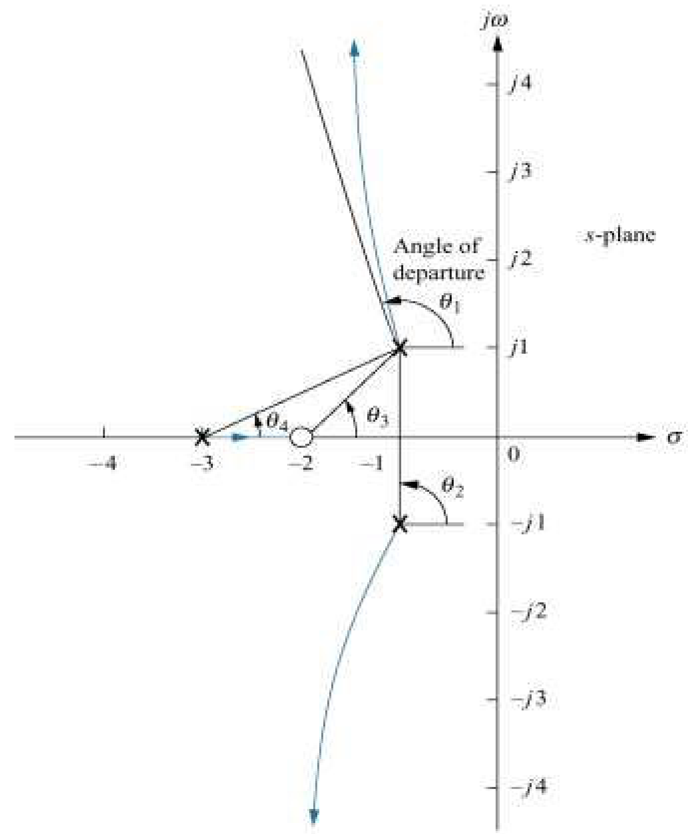
Example:
Given the unity feedback system, find the angle of departure from the complex poles & sketch the root locus



$$KG(s)H(s) = \frac{K(s+2)}{(s+3)(S^2+2s+2)}, H(s) = 1$$

where

$$KG(s)H(s) = \frac{K(s+2)}{(s+3)(s+1-j1)(s+1+j1)}$$



Root locus for the system showing angle of departure

Continuation of Previous Problem Solving

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = (2k+1)180^0 = 180^0 \ (k=0)$$

$$-\theta_1 - 90^0 + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^0$$

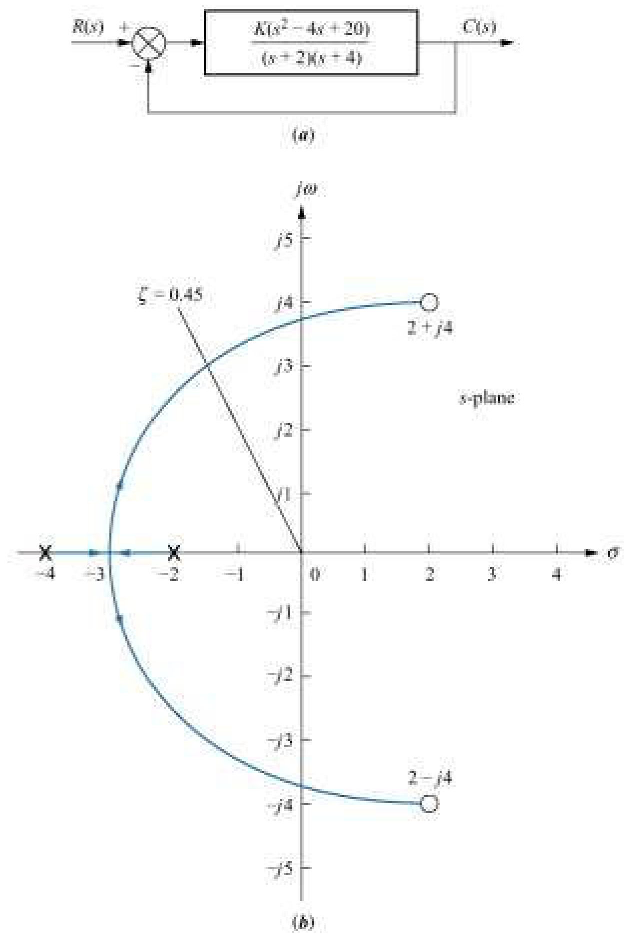
$$\theta_1 = -90^0 + 45^0 - 26.5^0 - 180^0$$

$$= -251.6^0 = 108.4^0$$

$The\ angle\ of\ departure\ of\ the\ complex\ pole\ is\ -108.4^0$
$(symmetry\ about\ the\ real\ axis)$

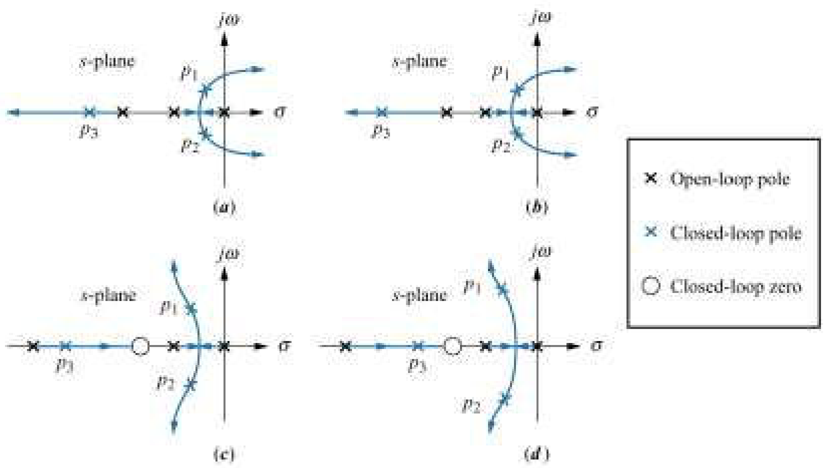
An Example

- Sketching the root locus & Finding the critical points
  - Find the exact point and gain where the locus crosses the 0.45 damping ratio line
  - Find the exact point and gain where the locus crosses the jw-axis
  - Find the breakaway point on the real axis
  - Find the range K within which the system is stable



Transient Response Design via Gain Adjustment

- Use Second order approximation which satisfy the following conditions:
  – Higher order poles are much farther into the left half of the s-plane than the dominant second order pair of poles. The response that results from a higher order pole does not appreciably change the transient response expected from the dominant second order poles
  – Closed loop zeros near the closed loop second order pole pair are nearly cancelled by the close proximity of higher order closed loop poles
  – Closed loop zeros not cancelled by the close proximity of higher order closed loop poles are far removed from the closed loop second order pole pair



Pole Sensitivity

- Since Root Locus is a plot of the Closed Loop Poles as a system parameter is varied → any change in the parameter will change the system performance too!
- Root Locus exhibits nonlinear relationship between gain and pole
  - Along some sections of the RL – very small changes in gain yield very large changes in pole location and hence performance → High Sensitivity to changes in gain
  - Along other sections of the RL – very large changes in gain yield very small changes in pole location → Low Sensitivity to changes in gain
- Preferences → Low Sensitivity to changes in gain

References

- www.slideshare.net
- www.nptel.ac.in