Numerical Methods to Solve ODE of First Order

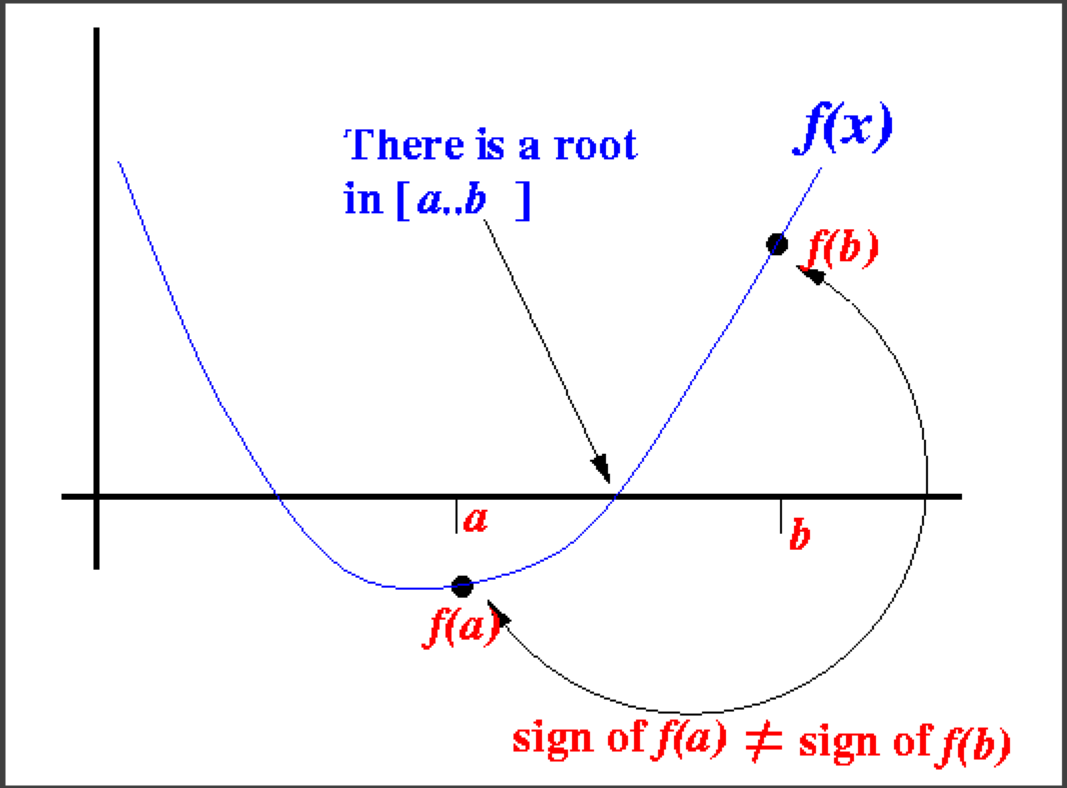
Bisection Method

Newton – Raphson method

Bisection Method

If a **function *f(x)*** is **continuous** on the **interval** [*a*. *b*] and **sign of *f(a)* ≠ sign of *f(b)***, then:

There is a value ***c*** **∈ [*a*. *b*]** such that: ***f(c) = 0*** I.e., there is a **root *c*** in the **interval [*a*. *b*]**



Bisection Method

•The **Bisection Method** is a ***successive* approximation method** that **narrows down** an **interval** that contains a **root of the function *f(x)***

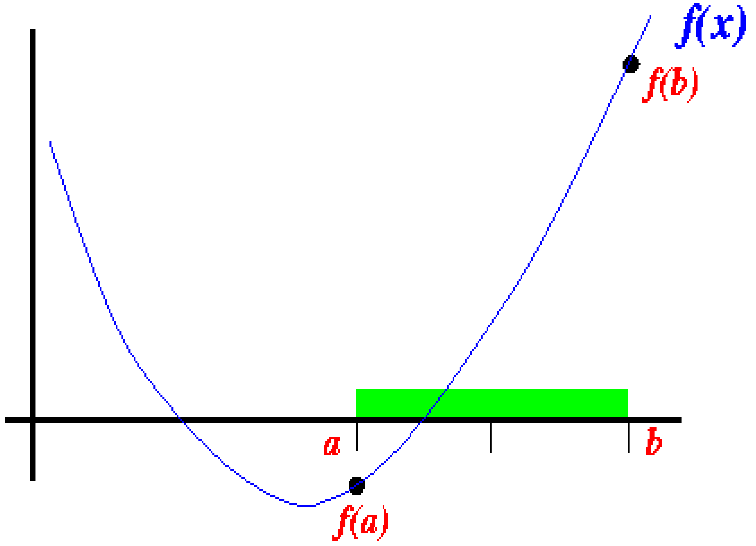
•The **Bisection Method** is ***given*** an **initial interval [*a*, *b*]** that **contains a root**
(We can use the **property** **sign of *f(a)* ≠ sign of *f(b)*** to find such an **initial interval**)

•The **Bisection Method** will ***cut the interval*** into **2 halves** and check **which half interval** contains a **root of the function**

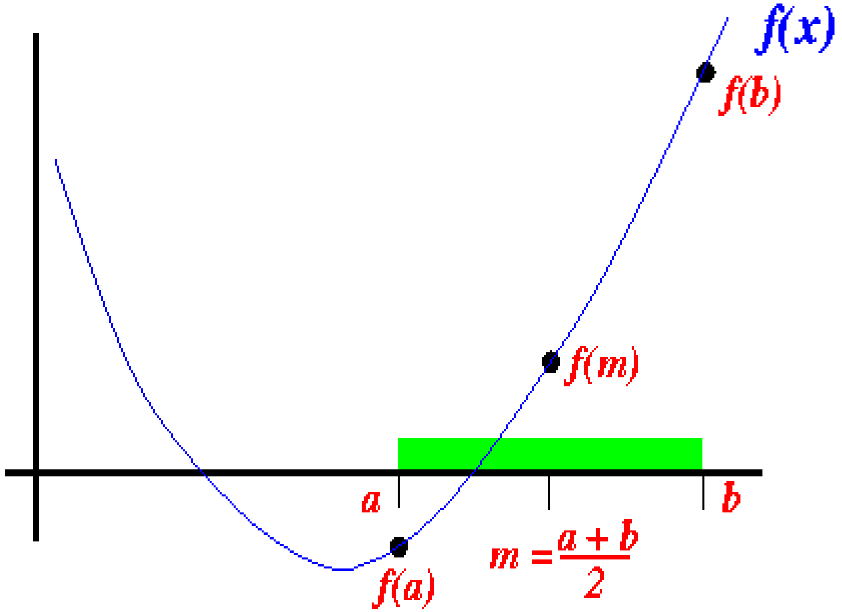
•The **Bisection Method** will keep ***cut the interval*** in halves until the **resulting interval** is **extremely small**

The **root** is then ***approximately equal*** to ***any value*** in the **final (very small) interval**.

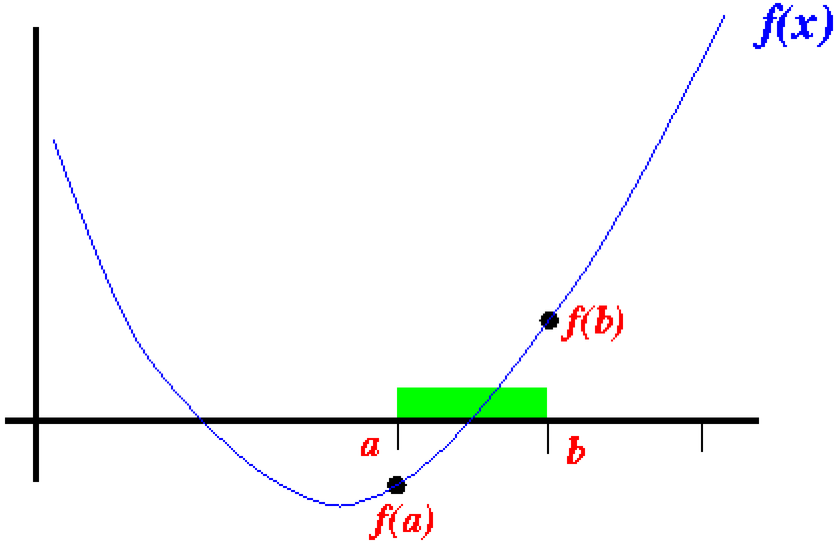
Suppose the **interval** [*a*, *b*] is as follow:



We **cut** the **interval [*a*..*b*]** in the **middle**: ***m* = (*a*+*b*)/2**



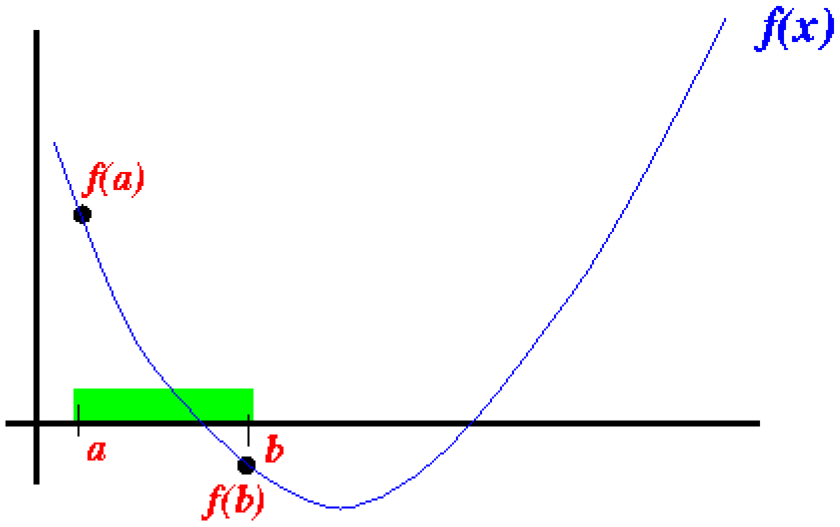
Because **sign of *f(m)* ≠ sign of *f(a)*** , we ***proceed*** with the **search** in the ***new* interval [*a, b*]**



So, we have **changed the end point *b*** to obtain a **smaller interval** that still contains a **root**

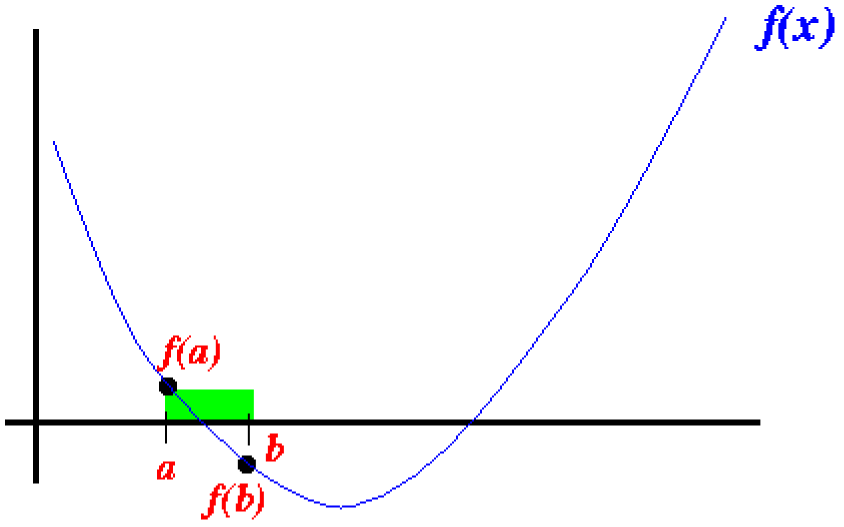
Now, we have an example to change the **end point *a***:

Initial **interval [*a, b*]**:



After **cutting the interval** in **half**, the **root** is contained in the **right-half**

So, we have to change the **end point *a***



Example: Find the root of ***f(x) = x² – 5*** between **[0, 4]**

There is a root between [0,4] because::

f(0) = 02 – 5 = –5
f(4) = 42 – 5 = 11

Start:

a = 0; f(a) = -5
b = 4; f(b) = 11

Iteration 1:

m = (a + b)/2 = 2

f(m) = 22 – 5 = -1

Because f(m) < 0, we replace a with m

a = 2; f(a) = -1
b = 4; f(b) = 11

Iteration 2:

m = (a + b)/2 = 3

f(m) = 32 – 5 = 4

Because f(m) > 0, we replace b with m

a = 2; f(a) = -1
b = 3; f(b) = 4

Iteration 3:

m = (a + b)/2 = 2.5

f(m) = 2.52 – 5 = 1.25

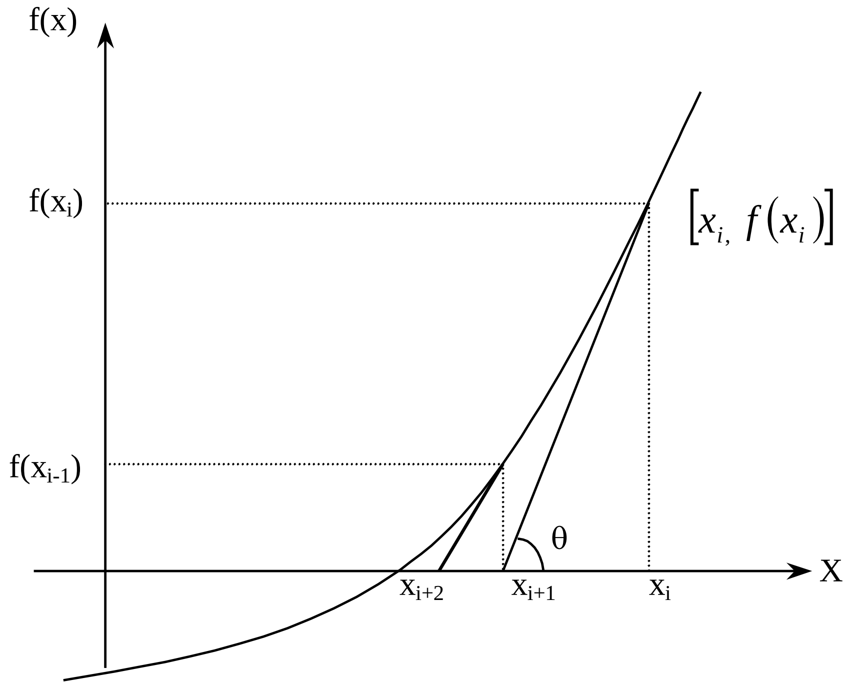
Because f(m) > 0, we replace b with m

a = 2; f(a) = -1
b = 2.5; f(b) = 1.25

And so on....

Approximate solution = 1.7320518493652344

Newton - Raphson's Method



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Geometrical illustration of the Newton-Raphson method

Evaluate $f'(x)$ symbolically.

Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Find the absolute relative approximate error $\left|\in_a\right|$ as

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Example 1:

Solve for $f'(x)$

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

$$f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of $f(x) = 0$ is $x_0 = 0.05\text{m}$. This is a reasonable guess (discuss why $x = 0$ and $x = 0.11\text{m}$ are not good choices) as the extreme values of the depth x would be 0 and the diameter (0.11 m) of the ball.

Example 1 Cont.

<u>Iteration 1</u>

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

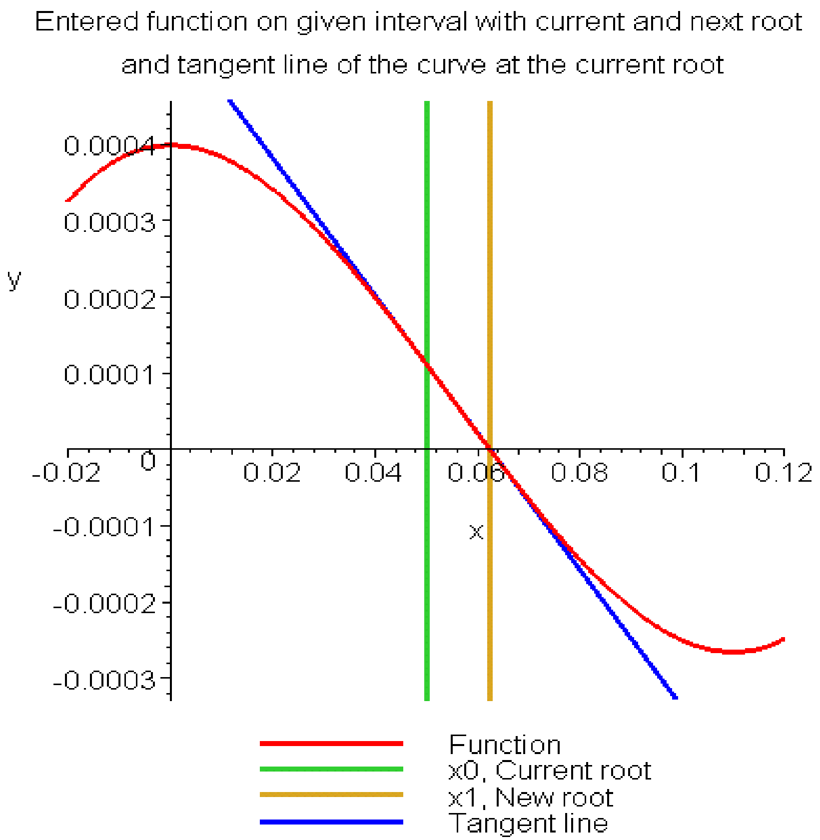
$$= 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993\times10^{-4}}{3(0.05)^2 - 0.33(0.05)}$$

$$= 0.05 - \frac{1.118\times10^{-4}}{-9\times10^{-3}}$$

$$= 0.05 - (-0.01242)$$

$$= 0.06242$$

Example 1 Cont.



Example 1 Cont.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06242 - 0.05}{0.06242}\right| \times 100$$

$$= 19.90\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for at least one significant digits to be correct in your result.

Example 1 Cont.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

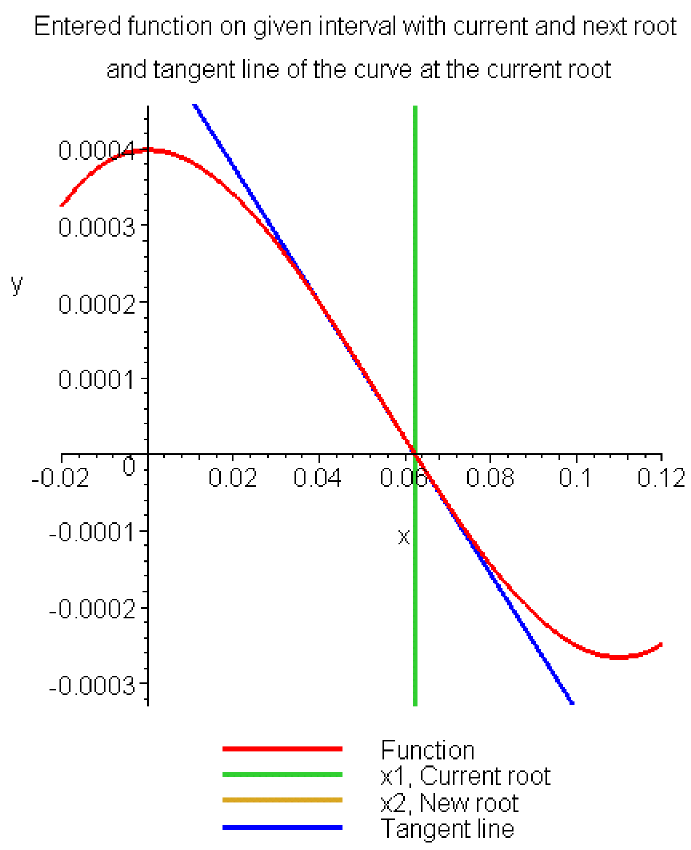
$$= 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993\times10^{-4}}{3(0.06242)^2 - 0.33(0.06242)}$$

$$= 0.06242 - \frac{-3.97781\times10^{-7}}{-8.90973\times10^{-3}}$$

$$= 0.06242 - \left(4.4646\times10^{-5}\right)$$

$$= 0.06238$$

Example 1 Cont.



Estimate of the root for the Iteration 2.

Example 1 Cont.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06242}{0.06238}\right| \times 100$$

$$= 0.0716\%$$

The maximum value of m for which $|\in_a| \leq 0.5 \times 10^{2-m}$ is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

Example 1 Cont.

<u>Iteration 3</u>

The estimate of the root is

$$x_3 = x_2 - \frac{f\left(x_2\right)}{f\ '\left(x_2\right)}$$

$$= 0.06238 - \frac{\left(0.06238\right)^3 - 0.165\left(0.06238\right)^2 + 3.993\times10^{-4}}{3\left(0.06238\right)^2 - 0.33\left(0.06238\right)}$$

$$= 0.06238 - \frac{4.44\times10^{-11}}{-8.91171\times10^{-3}}$$

$$= 0.06238 - \left(-4.9822\times10^{-9}\right)$$

$$= 0.06238$$

Example 1 Cont.

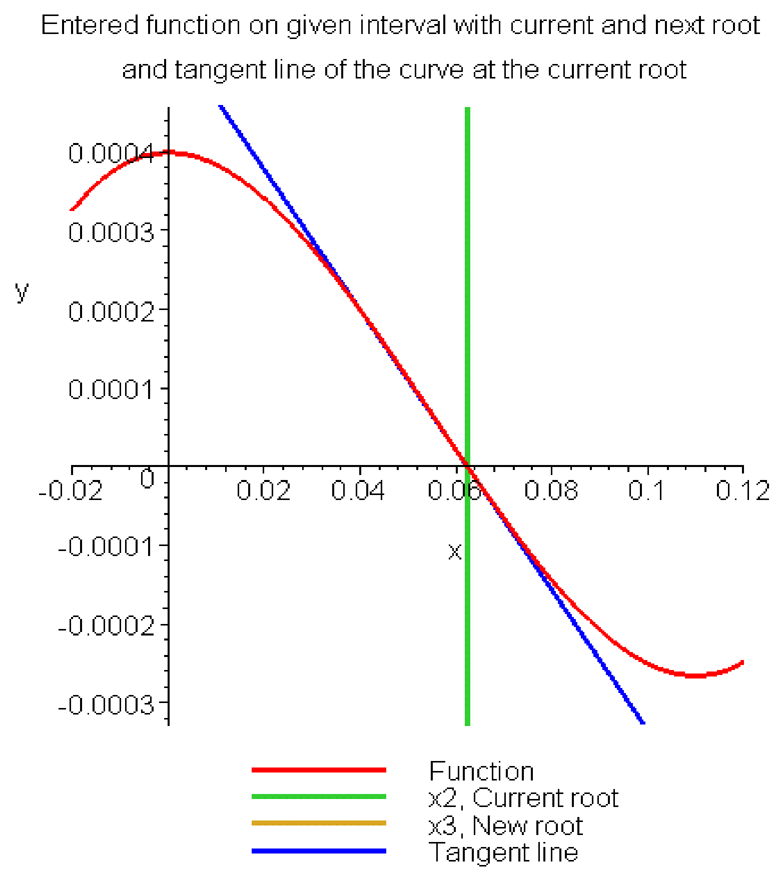


Fig: Estimate of the root for the Iteration 3.

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06238}{0.06238}\right| \times 100$$

$$= 0\%$$

The number of significant digits at least correct is 4, as only 4 significant digits are carried through all the calculations.

Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

Drawbacks-<u>Divergence at inflection points</u>

1. **<u>Divergence at inflection points:</u>**
   Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in the Newton-Raphson method.

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$.

The Newton-Raphson method reduces to $x_{i+1} = x_i - \dfrac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$.

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2.$

Drawbacks – Division by Zero

2. **Division by zero**

For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.

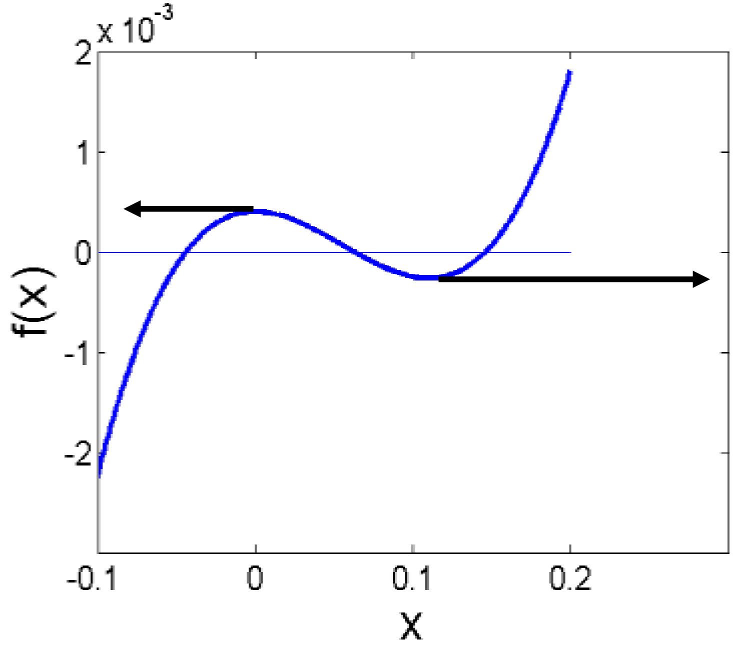


Fig: Pitfall of division by zero or near a zero number

Drawbacks – Inflection Points

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

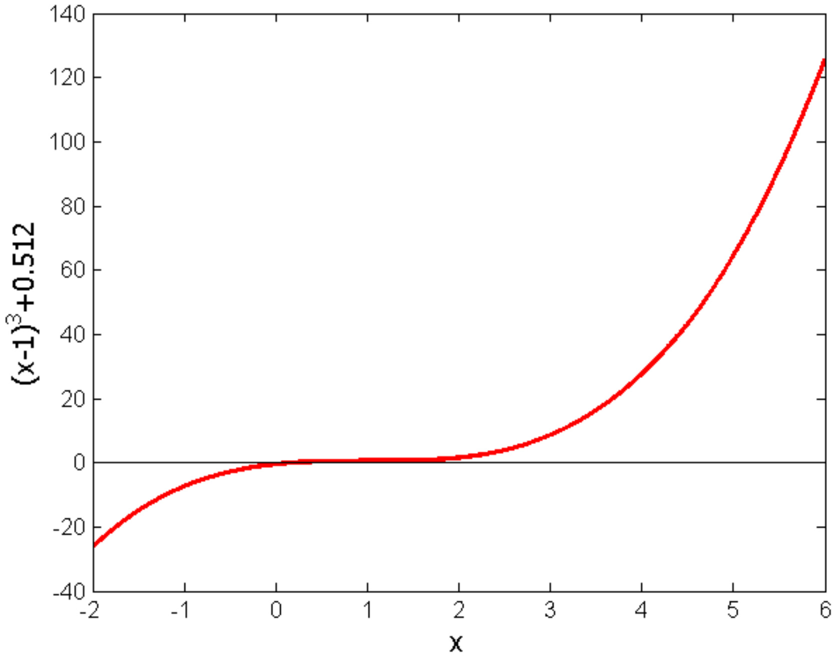


Fig: Divergence at inflection point for

$$f(x) = (x-1)^3 + 0.512 = 0$$

Drawbacks – Oscillations near local maximum and minimum

3. Oscillations near local maximum and minimum

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero and may diverge.

For example for $f(x) = x^2 + 2 = 0$ the equation has no real roots.

Drawbacks – Oscillations near local maximum and minimum

Table 3 Oscillations near local maxima and mimima in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	–1.0000	3.00	
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

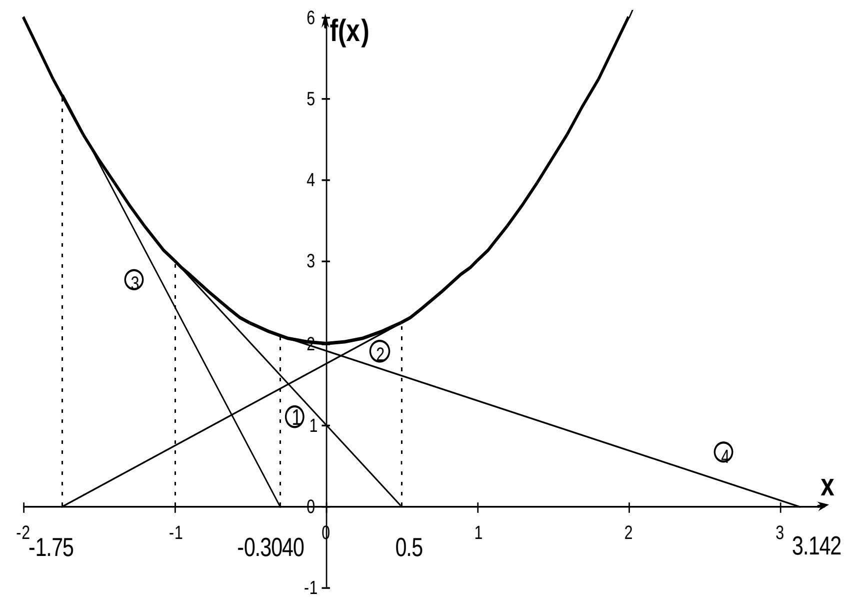


Fig: Oscillations around local minima for $f(x) = x^2 + 2$

Drawbacks – Root Jumping

4. <u>Root Jumping</u>

In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example

$$f(x) = \sin x = 0$$

Choose

$$x_0 = 2.4\pi = 7.539822$$

It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$

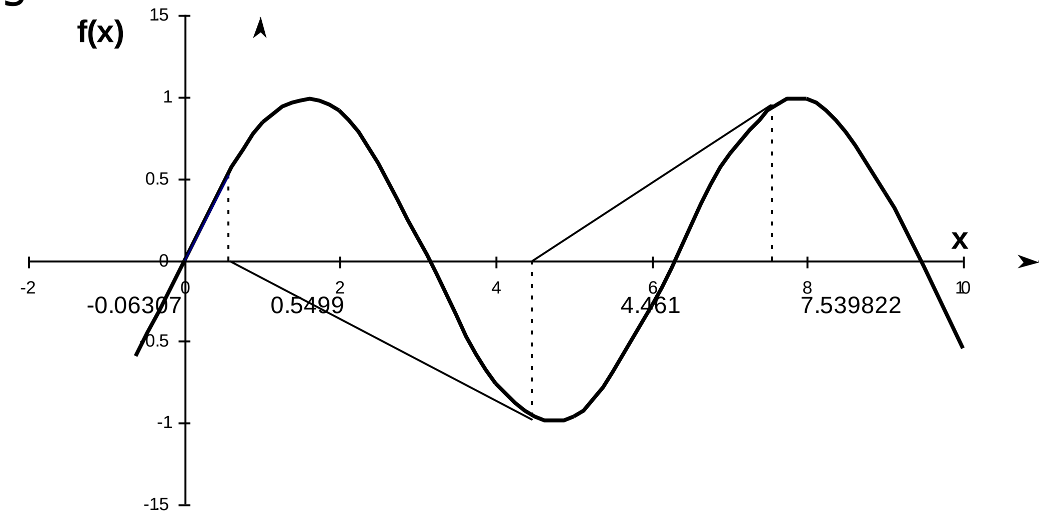


Fig: Root jumping from intended location of root for $f(x) = \sin x = 0$

.

Ordinary Differential Equations

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib

Syllabus

The syllabus contains the following articles:

- First Order Differential Equation
  - Leibnitz linear equation
  - Bernoulli's equation
  - Exact differential equation
  - Equations not of first degree
    - Equation solvable for p
    - Equation solvable for x
    - Equation solvable for y
  - Clairaut's equation
- Higher Order Differential Equation
  - Second order linear differential equations with variable coefficients
  - Method of variation of parameters
  - Power series solutions

Leibnitz linear equation

Definition

An equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are either constants or functions of x only is called Leibnitz linear equation.
Alternately, the equation may be of the form $\frac{dx}{dy} + Px = Q$, where P and Q are either constants or functions of y only.

Solution

This equation is solved by evaluating the Integration Factor that is given by $IF = e^{\int Pdx}$ and the solution is obtained by $y(IF) = \int Q(IF)dx + c$ for the former case and for the latter x is replaced by y in the IF and the solution.

Questions

- $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$
- $x \log x \dfrac{dy}{dx} + y = 2 \log x$
- $\dfrac{dy}{dx} + y \cot x = 5e^{\cos x}$
- $\dfrac{dy}{dx} = \dfrac{y}{2y \log y + y - x}$
- $\sqrt{1 - y^2} dx = (\sin^{-1} y - x) dy$

Bernoulli's Equation

Definition

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are either constants or functions of x only is called Bernoulli's equation.
Alternately, the equation may also be written as $\frac{dx}{dy} + Px = Qx^n$, where P and Q are either constants or functions of y only.

Solution

This equation is reduced to Leibnitz linear equation by substituting $y^{1-n} = z$ and differentiating. This generates the Leibnitz equation in z and x that is solved as explained earlier and then z is resubstituted in terms of y. The corresponding changes are made in the latter case of definition.

Questions

- $x\dfrac{dy}{dx} + y = x^3y^6$
- $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$
- $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$
- $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$
- $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

Exact Differential Equation

Definition

An equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be an Exact differntial equation if it can be obtained directly by differentiating the equation $u(x,y) = c$, which is its primitive.
i.e. if

$$du = Mdx + Ndy$$

Necessary and Sufficient Condition

The necessary and sufficient condition for the equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

The solution of $Mdx + Ndy = 0$ is given by

$$\int_{y \text{ constant}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$$

Questions

- $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$
- $(1 + e^{x/y})dx + \left(1 - \dfrac{x}{y}\right) e^{x/y} dy = 0$
- $(2xy \cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$
- $xdy + ydx + \dfrac{xdy - ydx}{x^2 + y^2} = 0$
- $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

Equations reducible to exact equations

Reducible to exact equations

Equations which are not exact can sometimes be made exact after multiplying by a suitable factor (function of x and/or y) called the Integration Factor (IF).

IF by Inspection

- $ydx + xdy = d(xy)$
- $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$
- $\dfrac{xdy - ydx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
- $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right]$
- $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
- $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$
- $\dfrac{ydx + xdy}{xy} = d[\log(xy)]$
- $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

Equations reducible to exact equations

IF for Homoegeneous Equation

If $Mdx + Ndy = 0$ is a Homoegeneous equation in x and y, then $\dfrac{1}{Mx + Ny}$ is an IF provided $Mx + Ny \neq 0$.

IF for $f_1(xy)ydx + f_2(xy)xdy = 0$

For equation of this type, IF is given by $\dfrac{1}{Mx - Ny}$.

Equations reducible to exact equations

IF for $Mdx + Ndy = 0$

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $IF = e^{\int f(x)dx}$.
- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say $g(y)$, then $IF = e^{\int g(y)dy}$.

IF for $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$

In this equation, a, b, c, d, m, n, p, q are all constants and IF is given by $x^h y^k$, where h and k are so chosen that the equation becomes exact after multiplication with IF.

QUESTIONS

- $(1+xy)ydx + (1-xy)xdy = 0$
- $xdy - ydx = xy^2dx$
- $(xye^{x/y} + y^2)dx - x^2e^{x/y}dy = 0$
- $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$
- $\left(y + \dfrac{y^3}{3} + \dfrac{x^2}{2}\right) dx + \dfrac{1}{4}(x + xy^2)dy = 0$
- $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$
- $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

Equations of first order and higher degree

Definition

A differential equation of the first order and n^{th} degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0, \text{ where } p = \frac{dy}{dx} \tag{1}$$

Equations solvable for p

Resolve equation (1) into n linear factors and solve each of the factors to obtain solution of the given equation.

Questions

- $p^2 - 7p + 12 = 0$
- $xyp^2 - (x^2 + y^2)p + xy = 0$
- $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$
- $p^2 - 2p\sinh x - 1 = 0$
- $4y^2p^2 + 2pxy(3x+1)3x^3 = 0$

Equations solvable for y

Differentiate equation (1), wrt x, to obtain a differential equation of first order in p and x that has solution of the form $\phi(x,p,c)=0$. The elimination p from this solution and equation (1) gives the desired solution.

Questions

- $xp^2 - 2yp + ax = 0$
- $y - 2px = \tan^{-1}(xp^2)$
- $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$
- $x - yp = ap^2$

Equations solvable for x

Differentiate equation (1), wrt y, to obtain a differential equation of first order in p and y that has solution of the form $\phi(y,p,c)=0$. The elimination p from this solution and equation (1) gives the desired solution.

Questions

- $y=3px+6p^2y^2$
- $p^3-4xyp+8y^2=0$
- $y=2px+p^2y$
- $y^2\log y=xyp+p^2$

Clairaut's equation

Definition

An equation of the form $y = px + f(p)$ is called Clairaut's equation.

Solution

Differente the equation wrt x, and obtain the solution by putting $p = c$ in the given equation.

Questions

- $y = xp + \dfrac{a}{p}$
- $y = px + \sqrt{a^2p^2 + b^2}$
- $p = \sin(y - px)$
- $p = \log(px - y)$

Linear Differential Equations

Definition

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \tag{2}$$

Linear Differential Equations

Complementary Function (CF)

- If all the roots of equation (2) are real and distint, CF is given by $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
- If two roots are equal, say $m_1 = m_2$, then CF is given by $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
- If two roots are imaginary, say $m_1 = \alpha + \iota\beta,\ m_2 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
- It two pairs of imaginary roots are equal, say $m_1 = m_2 = \alpha + \iota\beta,\ m_3 = m_4 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

Linear Differential Equations

Particular Integral (PI)

- If $X = e^{ax}$, then PI is given by $y = \dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$.
- If $X = \sin(ax+b)$ or $\cos(ax+b)$, then PI is given by $y = \dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)$. Likewise for $\cos(ax+b)$.
- If $X = x^m$, where m is a positive integer, then PI is given by $y = \dfrac{1}{(D)}x^m$.

  Take out the lowest degree term from $f(D)$ to make the first term unity and then shift the remaining term to numerator and apply Binomial expansion upto D^m. Operate term by term on x^m.
- If $X = e^{ax}V$, where V is a function of x, then PI is given by $y = \dfrac{1}{f(D)}e^{ax}V = e^{ax}\dfrac{1}{f(D+a)}V$.
- If X is any other function of x, then PI is obtained by resolving the $f(D)$ into linear factors and applying $\dfrac{1}{D-a}X = e^{ax}\displaystyle\int e^{-ax}X\,dx$

QUESTIONS

- $(D^2 + 4D + 5)y = -2\cosh x$
- $(D2 - 4D + 3)y = \sin 3x \cos 2x$
- $(D^2 + 4)y = e^x + \sin 2x$
- $(D^2 + D)y = x^2 + 2x + 4$
- $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$
- $(D^2 - 4D + 4)y = 8x^2e^{2x} \sin 2x$
- $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$
- $(D - 1)^2(D + 1)^2y = \sin^2 \frac{x}{2} + e^x + x$

Cauchy's Homogeneous Equation

Definition

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \qquad (3)$$

where a_is are constants and X is a function of x is called Cauchy's Homegeneous Linear Equation.

Solution

The equation is reduced to an LDE with constant coefficients by putting $z = e^x$ thereby generating an LDE in x and z that can be solved as explained earlier and finally the solution of equation (3) is obtained by putting $z = \log x$.

Questions

- $x^2\dfrac{d^2y}{dx^2}+9x\dfrac{dy}{dx}-25y=50$
- $x^4\dfrac{d^3y}{dx^3}+2x^3\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=1$
- $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=\dfrac{12\log x}{x^2}$
- $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+y=\log x\dfrac{\sin(\log x)+1}{x}$

Legendre's Linear Equation

Definition

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X \quad (4)$$

where a_is, a and b are constants and X is a function of x is called Legendre's Linear Equation.

Solution

The equation is reduced to an LDE with constant coefficients by putting $a+bx=e^z$ thereby generating an LDE in x and z that can be solved as explained earlier and finally the solution of equation (4) is obtained by putting $z=\log(a+bx)$.

Questions

- $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos\log(1+x)$
- $(1+2x)^2\dfrac{d^2y}{dx^2}-6(1+2x)\dfrac{dy}{dx}+16y=8(1+2x)^2$
- $(3+2x)^2\dfrac{d^2y}{dx^2}-2(3+2x)\dfrac{dy}{dx}-12y=6x$

Variation of Parameters

This method is applicable for the second order differential equation of the form $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = X$
Let the CF of this equation be

$$y = c_1y_1 + c_2y_2$$

. Then the PI of this equation is given by

$$y = uy_1 + vy_2$$

where

$$u = -\int \frac{y_2X}{W}dx$$

and

$$v = \int \frac{y_1X}{W}dx$$

where W is the Wronskian of $y_1,\ y_2$.

QUESTIONS

- $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$
- $\dfrac{d^2y}{dx^2} + y = cosec\ x$
- $\dfrac{d^2y}{dx^2} + y = x\sin x$
- $y'' - 2y' + 2y = e^x \tan x$

Series Solution

BBSBEC

We discuss the method of solving equations of the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{5}$$

where $P_0(x), P_1(x)$ and $P_2(x)$ are polynomials in x, in terms of infinite convergent series.

Solution

Divide equation (5) by $P_0(x)$ to get

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{6}$$

where $p(x) = \dfrac{P_1(x)}{P_0(x)}$ and $q(x) = \dfrac{P_2(x)}{P_0(x)}$

Series Solution

Ordinary Point

$x = 0$ is called an ordinary point of equation (5) if $P_0(0) \neq 0$.
In this casem the solution of equation (5), can be expressed as

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Singular Point

$x = 0$ is called a singular point of equation (5), if $P_0(0) = 0$.
In this case, the solution of equation (5) can be expressed as

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Solution when $x = 0$ is an ordinary point

Solution

Let $y = \sum_{k=0}^{\infty} a_k x^k$ be the solution of equation (5). Then, on differentiating

$$\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

1. Substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (5).
2. Equate to zero the coefficients of various powers of x and find $a_2, a_3, a_4, \ldots$ in terms of a_0 and a_1.
3. Equate to zero the coefficient of x^n. The relation so obtained is called the recurrence relation.
4. Give different values to n in the recurrence relation to determine various a_is in terms of a_0 and a_1.
5. Substitute the values in the above mentioned series to obtain the solution with a_0 and a_1 as arbitrary constants.

QUESTIONS

- $\dfrac{d^2y}{dx^2} + xy = 0$
- $y'' - xy' + x^2y = 0$
- $(2 - x^2)y'' + 2xy' - 2y = 0$

Solution when $x = 0$ is a regular singular point I

Let $y = \sum_{k=0}^{\infty} a_k x^{m+k}$ be the solution of equation (5). Then, on differentiating

$$\frac{dy}{dx} = \sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}.$$

1. Substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (5).
2. Equate to zero the coefficients of lowest powers of x. This gives a quadratic equation in m, which in known as indicial equation.
3. Equate to zero the coefficients of other powers of x to find $a_1, a_2, a_3, a_4, \ldots$ in terms of a_0.
4. Substitute the values of $a_1, a_2, a_3, \ldots$ in above said solutionto get the series solution of (5) having a_0 as the arbitrary constant. Though, it is not the complete solution as the same should have two arbitrary constants.
5. The method of complete solution depends on the nature of roots of the indicial equation.

Solution when $x = 0$ is a regular singular point II

Case I When the roots m_1, m_2 are distinct and not differing by an integer. Then the complete solution is given by

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

Case II When the roots m_1, m_2 are equal. Then the complete solution is given by

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m_1}$$

Case III When the roots $m_1 < M_2$ are distinct and differ by an integer. Then th ecomplete solution is given by

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m_1}$$

QUESTIONS

- $2x^2\dfrac{d^2y}{dx^2} + (2x^2 - x)\dfrac{dy}{dx} + y = 0$
- $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 4)y = 0$
- $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0$