Subject: Mathematics-II Code: BTAM201-18

Branch: Civil Engineering Semester: 2nd

SECTION – B

Unit-III & IV

Partial Differential Equations: Higher Order

Introduction
Partial Differential Equations

- Partial Differential Equations (PDEs).
- What is a PDE?
- Examples of Important PDEs.
- Classification of PDEs.

Partial Differential Equations

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

Example :

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

PDE involves two or more independent variables

(in the example x and t are independent variables)

Notation

$$u_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2 u(x,t)}{\partial x \, \partial t}$$

Order of the PDE = order of the highest order derivative.

Linear PDE

Classification

A PDE is linear if it is linear in the unknown function and its derivatives

Example of linear PDE :

$$2\,u_{xx} + 1\,u_{xt} + 3\,u_{tt} + 4\,u_x + \cos(2t) = 0$$

$$2\,u_{xx} - 3\,u_t + 4\,u_x = 0$$

Examples of Nonlinear PDE

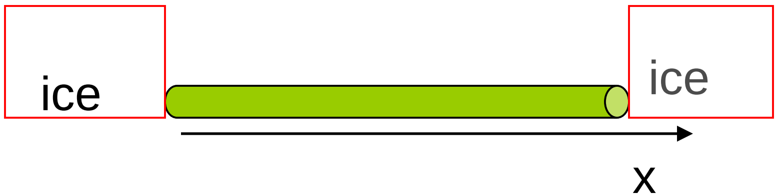
$$2\,u_{xx} + \underline{(u_{xt})^2} + 3\,u_{tt} = 0$$

$$\underline{\sqrt{u_{xx}}} + 2\,u_{xt} + 3u_t = 0$$

$$2\,u_{xx} + \underline{2\,u_{xt}\,u_t} + 3u_t = 0$$

Heat Equation

Different curve is used for each value of t

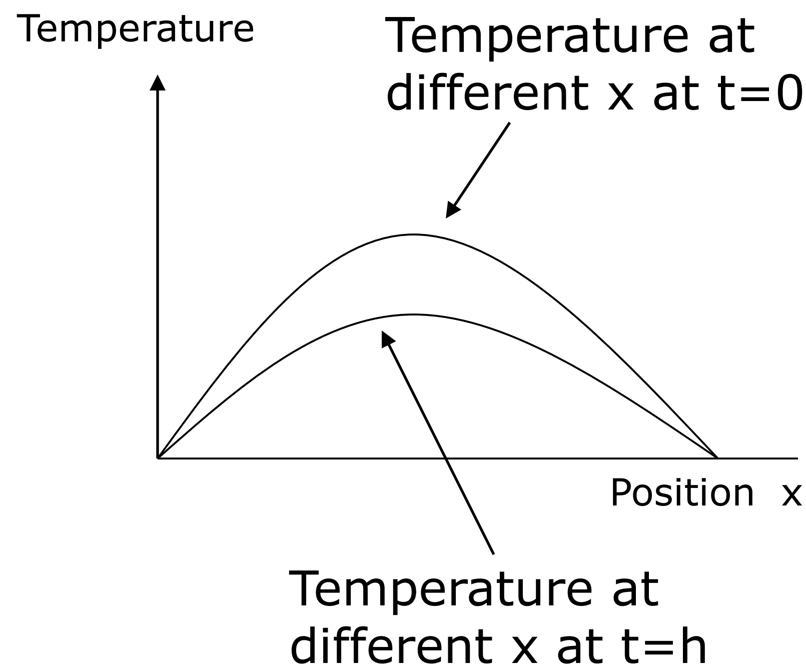


Thin metal rod insulated everywhere except at the edges. At t =0 the rod is placed in ice

$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,0) = \sin(\pi x)$$



Examples of PDEs

PDEs are used to model many systems in many different fields of science and engineering.

Important Examples:

- Laplace Equation
- Heat Equation
- Wave Equation

Laplace Equation

$$\frac{\partial^2 u(x,y,z)}{\partial x^2}+\frac{\partial^2 u(x,y,z)}{\partial y^2}+\frac{\partial^2 u(x,y,z)}{\partial z^2}=0$$

Used to describe the steady state distribution of heat in a body.

Also used to describe the steady state distribution of electrical charge in a body.

Heat Equation

$$\frac{\partial u(x,y,z,t)}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The function $u(x,y,z,t)$ is used to represent the temperature at time t in a physical body at a point with coordinates (x,y,z)

α is the thermal diffusivity. It is sufficient to consider the case $\boldsymbol{\alpha = 1}$.

Simpler Heat Equation

$$\frac{\partial\, T(x,t)}{\partial t} = \frac{\partial^2\, T(x,t)}{\partial x^2}$$

T(x,t) is used to represent the temperature at time *t* at the point *x* of the thin rod.

Wave Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

The function *u(x,y,z,t)* is used to represent the displacement at time t of a particle whose position at rest is *(x,y,z)* .

The constant *c* represents the propagation speed of the wave.

Classification of PDEs

Linear Second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.

Classification is important because:

- Each category relates to specific engineering problems.
- Different approaches are used to solve these categories.

Linear Second Order PDEs
Classification

A second order linear PDE (2 - independent variables)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of $x, y, u, u_x,$ and u_y

is classified based on $(B^2 - 4AC)$ as follows:

$B^2 - 4AC < 0$	Elliptic
$B^2 - 4AC = 0$	Parabolic
$B^2 - 4AC > 0$	Hyperbolic

Linear Second Order PDE
Examples (Classification)

$$\text{Laplace Equation} \quad \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1 \Rightarrow B^2 - 4AC < 0$$

$$\Rightarrow \text{Laplace Equation } \textit{is Elliptic}$$

$$\text{One possible solution: } u(x,y) = e^x \sin y$$

$$u_x = e^x \sin y, \quad u_{xx} = e^x \sin y$$

$$u_y = e^x \cos y, \quad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = 0$$

Linear Second Order PDE

Examples (Classification)

Heat Equation $\alpha \dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{\partial u(x,t)}{\partial t} = 0$

$A = \alpha,\ B = 0,\ C = 0 \Rightarrow B^2 - 4AC = 0$

$\Rightarrow$ Heat Equation *is Parabolic*

Wave Equation $c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{\partial^2 u(x,t)}{\partial t^2} = 0$

$A = c^2 > 0,\ B = 0,\ C = -1 \Rightarrow B^2 - 4AC > 0$

$\Rightarrow$ Wave Equation *is Hyperbolic*

Boundary Conditions for PDEs

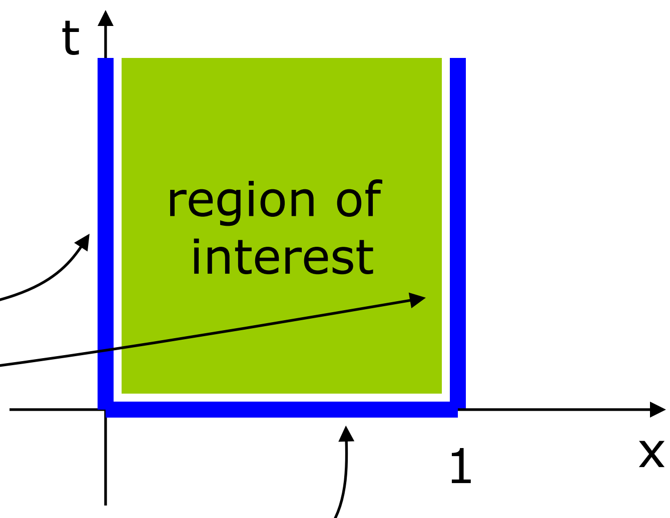
- To uniquely specify a solution to the PDE, a set of boundary conditions are needed.
- Both regular and irregular boundaries are possible.

$$\text{Heat Equation}: \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



Parabolic Equations

- Parabolic Equations
- Heat Conduction Equation
- Explicit Method
- Implicit Method
- Cranks Nicolson Method

Parabolic Equations

A second order linear PDE (2-independent variables x, y)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of x, y, u, u_x, and u_y

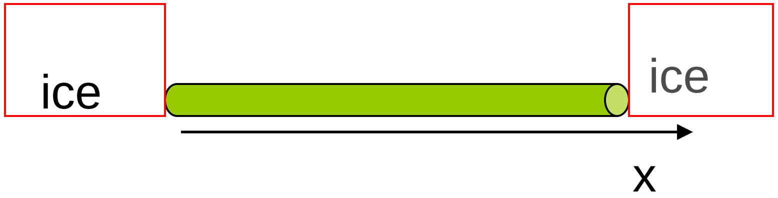
is parabolic if $\boxed{B^2 - 4AC = 0}$

Parabolic Problems

Heat Equation : $$\frac{\partial\, T(x,t)}{\partial t} = \frac{\partial^2\, T(x,t)}{\partial x^2}$$

$$\boxed{T(0,t) = T(1,t) = 0}$$

$$\boxed{T(x,0) = \sin(\pi\, x)}$$



* Parabolic problem $(B^2 - 4AC = 0)$

* Boundary conditions are needed to uniquely specify a solution.

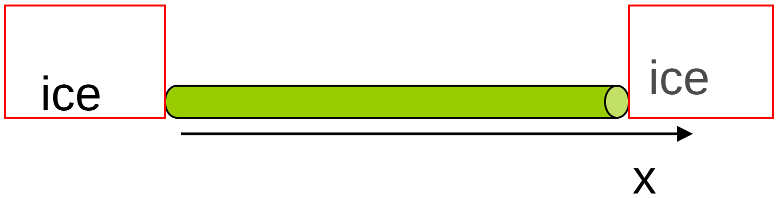
Example 1: Heat Equation

Solve the PDE :

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



$Use \quad h = 0.25, \quad k = 0.25$ to find $u(x,t)$ for $x \in [0,1], t \in [0,1]$

$$\lambda = \frac{k}{h^2} = 4$$

Elliptic Equations

- Elliptic Equations
- Laplace Equation
- Solution

Elliptic Equations

A second order linear PDE (2 - independent variables x , y)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of $x, y, u, u_x,$ and u_y

is Elliptic if $\boxed{B^2 - 4AC < 0}$

Laplace Equation

Laplace equation appears in several engineering problems such as:

- Studying the steady state distribution of heat in a body.
- Studying the steady state distribution of electrical charge in a body.

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

T : steady state temperature at point (x, y)

$f(x,y)$: heat source (or heat sink)

Laplace Equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

$$A = 1, B = 0, C = 1$$

$$B^2 - 4AC = -4 \quad < 0 \quad \textit{Elliptic}$$

- Temperature is a function of the position (x and y)
- When no heat source is available → $f(x,y)=0$

D'Alembert's Solution

There is an elegant approach to solve the wave equation by introducing new variables:

$$v = x + ct, \quad z = x - ct, \quad u(x,t) = u(v,z)\left\{ \quad \right\}$$

The use of these variables is because that the solution of the wave equation behaves in specific fashion that its spatial movement is related to the temporal variation through the constant *c*.

Using these new variables, the derivative w.r.t x & t can be rewritten as

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial (x+ct)}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial (x-ct)}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}$$

Label $u_x = \dfrac{\partial u}{\partial x}, u_v = \dfrac{\partial u}{\partial v}, u_z = \dfrac{\partial u}{\partial z}, etc$

Similarly,

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}}=\frac{\partial \mathrm{u}}{\partial v}\frac{\partial v}{\partial \mathrm{t}}+\frac{\partial \mathrm{u}}{\partial z}\frac{\partial \mathrm{z}}{\partial \mathrm{t}}=\frac{\partial \mathrm{u}}{\partial v}(c)+\frac{\partial \mathrm{u}}{\partial z}(-c)=c[\frac{\partial \mathrm{u}}{\partial v}-\frac{\partial \mathrm{u}}{\partial z}]$$

Continue to convert all derivatives in x & t into derivatives

in v & z, the wave equation to obtain the following equation:

$$\frac{\partial^2 u}{\partial z \partial v}=\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial v}\right)=0,$$ this equation can be integrated twice

$$\frac{\partial u}{\partial v}= f(v),$$

$$u(v,z)=\int f(v)dv+\psi(z)=\phi(v)+\psi(z)$$

$u(x,y)=\phi(x+ct)+\psi(x-ct)$: D'Alembert's solution

If given the initial conditions:

$$u(x,t=0)=f(x),\ \ \frac{\partial u}{\partial t}(x,t=0)=g(x)$$

Determine the D'Alembert's solution:

$$u(x,0)=\phi(x+ct)+\psi(x-ct)=\phi(x)+\psi(x)=f(x)$$

$$\frac{\partial u}{\partial t}(x,t)=\frac{\partial}{\partial v}[\phi(x+ct)]\frac{\partial v}{\partial t}+\frac{\partial}{\partial z}[\psi(x-ct)]\frac{\partial z}{\partial t}$$

$$=c\phi'(x+ct)-c\psi'(x-ct)$$

$$\frac{\partial u}{\partial t}(x,t=0)=c[\phi'(x)-\psi'(x)]=g(x)$$

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds$$

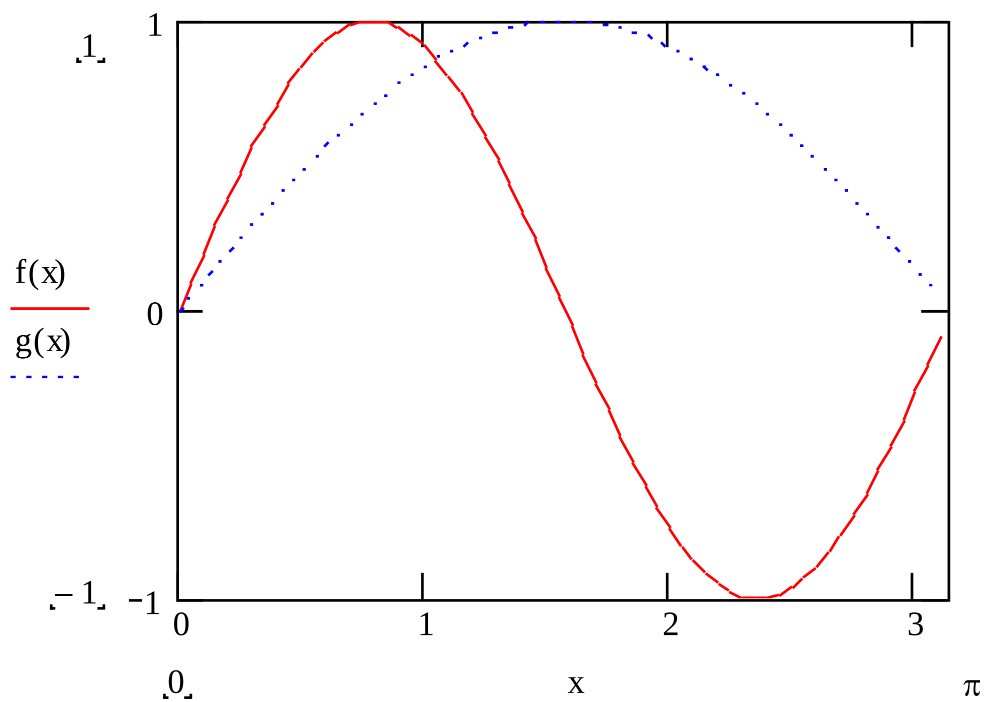
See textbook, chapter 11.4 for detailed derivation

Example

Determine the solution of the wave equation in d'Alembert form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad \text{for } 0 < x < \pi,\ t > 0,$$

$$u(x,0) = f(x) = \sin(2x), \frac{\partial u}{\partial t}(x,0) = g(x) = \sin(x)$$



From the wave equation, c=1

D'Alembert's solution:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

$$f(x+ct) = f(x+t) = \sin(2x+2t)$$

$$f(x-t) = \sin(2x-2t)$$

$$\int_{x-ct}^{x+ct} g(s)ds = \int_{x-t}^{x+t} \sin(s)ds = -\cos(s)\Big|_{x-t}^{x+t} = \cos(x-t) - \cos(x+t)$$

$$u(x,t) = \frac{1}{2}[\sin(2x+2t) + \sin(2x-2t)] + \frac{1}{2}\left[\cos(x-t) - \cos(x+t)\right]$$

A very simple form of solution, image how difficult it will be if one uses the separation of variables and Fourier series solution to solve this equation.

Wave Propagation

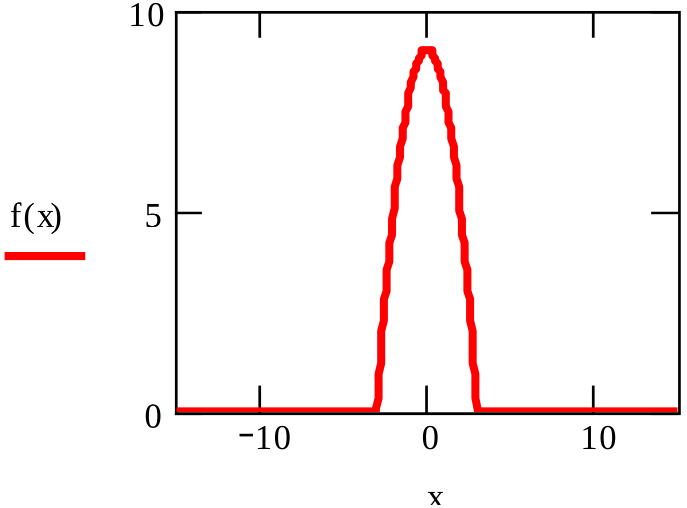
It is much easier to identify the wave propagation characteristics of the solution of the wave equation by examining an initial disturbance confined within a finite area and trace its development in time. Examine the solution when $c = 1,\ u(x,0) = f(x) = \begin{cases} 9 - x^2, \text{if } -3 \le x \le 3 \\ 0 \quad ,\text{Otherwise} \end{cases}$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = 0$$

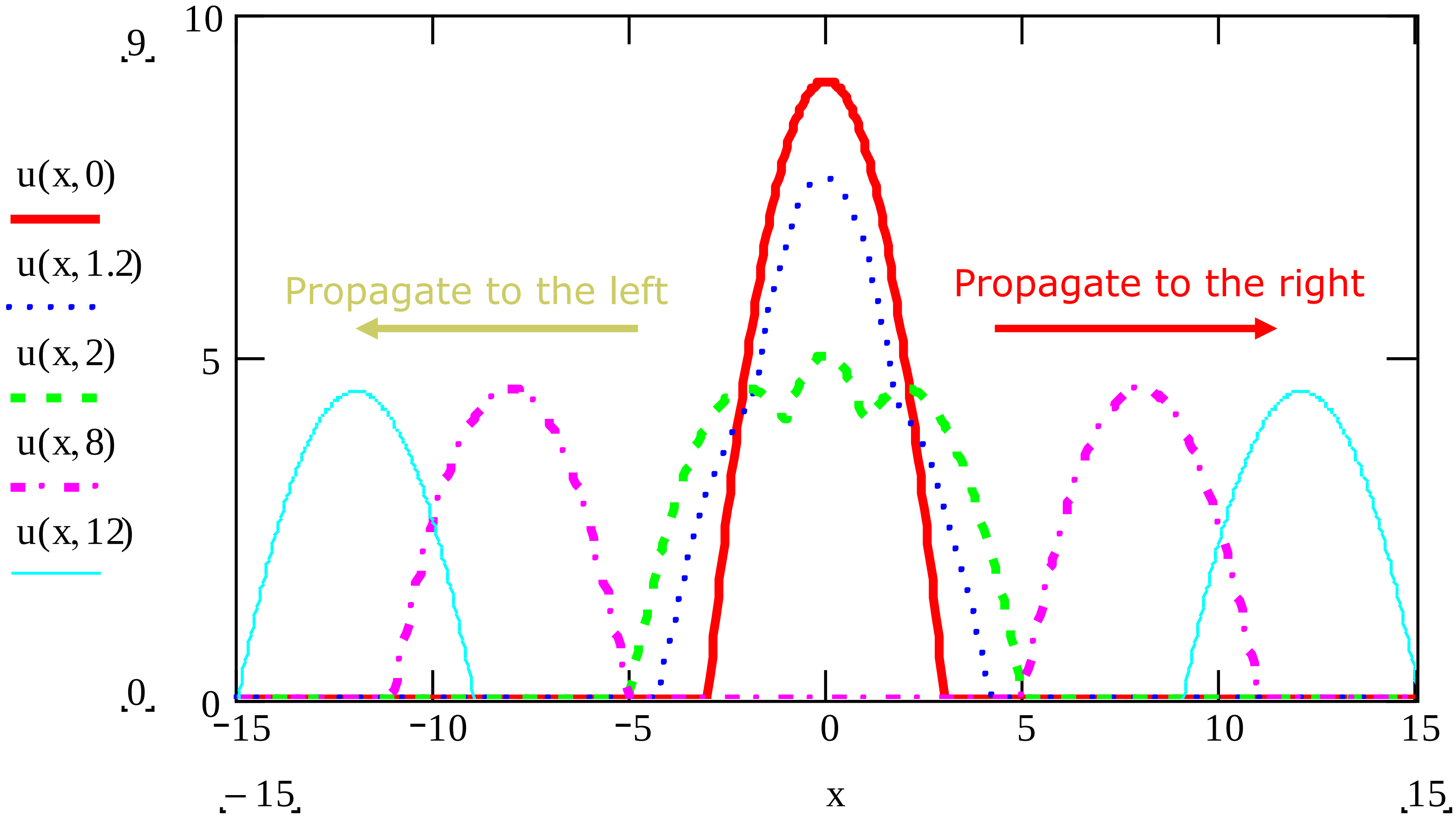
$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t))$$

As shown, the initiail disturbance can be interpreted as a pressure spike, a wave crest, a temperature surge, etc..

Initial condition



When t>0, the disturbance splits into two parts, one propagating to the right while the other propagating to the left, as shown below:



Characteristic Lines

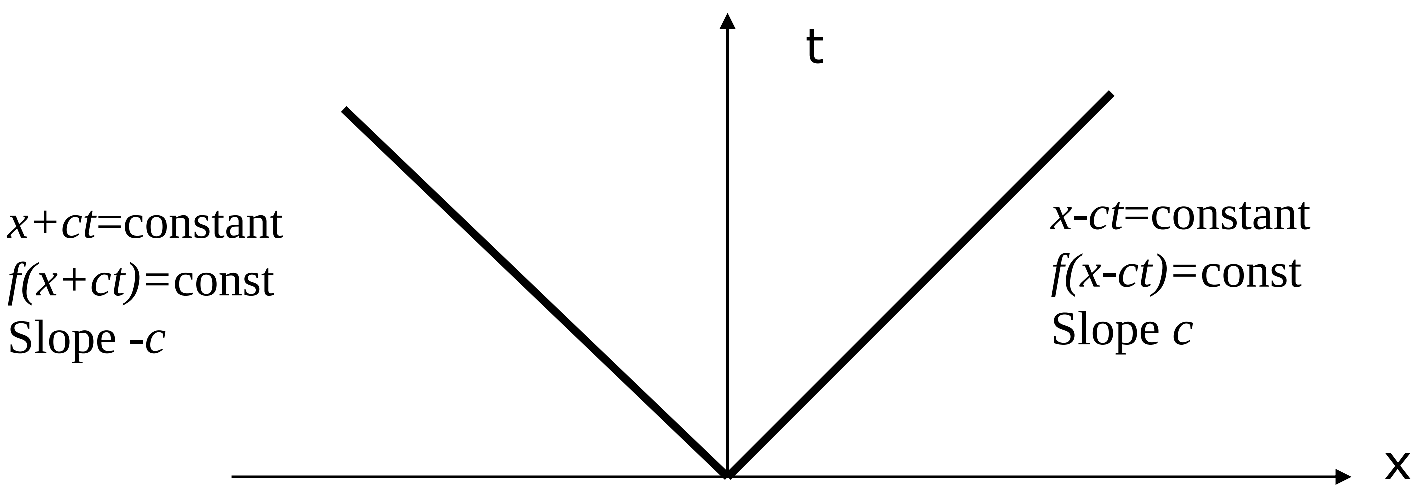
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x)$$

Assume $g(x) = 0$ for simplicity

D'Alembert's solution: $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

Specify $\xi = x + ct, \eta = x - ct$ therefore

$f(x+ct)$ remains the same as long as $\xi = x + ct$ remains a constant.



Nonhomogeneous: Wave Equation

Sometime we have deal with nonhomogeneous form of the wave equation.

For example, when the string or beam is under an external excitation, varying both in space and time: $F(x,t)$

The nonhomogeneous wave equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + F(x,t)$

with the initial conditions of : $u(x,0) = f(x), \dfrac{\partial u}{\partial t}(x,0) = g(x)$

It can be shown that (not here), the general solution if of the form

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\iint_{\Delta} F(s,\tau)dsd\tau$$

Δ is the characteristic triangle, a region in the x,t plane bounded by the two characteristic lines and the initial time line.

Separation of Variables Method With Applications

1. Introduction
2. Laplace's Equation
   - Steady-State temperature in a rectangular plate
   - Math. Parenthesis: The Fourier Series
3. The Diffusion or Heat Flow Equation
   - Flow of Heat through a slab of thickness ℓ
4. The Wave Equation
   - The Vibrating String

Introduction

Many of the problems of mathematical physics involve the solution of PDE's. The same PDE may apply to a variety of physical problems; then many of the methods will apply to a bigger set of problems.

- Laplace's Equation: $$\nabla^2 u = 0$$
- Poisson's Equation: $$\nabla^2 u = f(x, y, z)$$
- The difusion or heat flow equation: $$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$
- Wave equation: $$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Laplace Equation

Example (Steady-State temperature in a rectangular plate)

We have a rectangular plate on the region:
$R : \{0 < x < 10\,, \quad 0 < y\}$ with border conditions:
$T(x=0) = T(x=10) = T(y=\infty) = 0^\circ$ and $T(y=0) = 100^\circ$

Solution

$$\nabla^2 T = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(x,y) = X(x)Y(y)$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

Contd…

Solution

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = const. = -k^2$$

$$X'' = -k^2 X \quad and \quad Y'' = +k^2 Y$$

(soln's:)

$$X(x) = A\cos kx + B\sin kx \quad and \quad Y(y) = Ce^{ky} + De^{-ky}$$

Contd…

Solution

$$T(x,y) = X(x)Y(y) = (A\cos kx + B\sin kx)(Ce^{ky} + De^{-ky})$$

$$B.C.: T(x,\infty) = 0 \Rightarrow C = 0$$

$$T(0,y) = 0 \Rightarrow A = 0 \quad and$$

$$T(10,y) = 0 \Rightarrow \sin 10k = 0 \Rightarrow k = n\frac{\pi}{10})$$

$$\Rightarrow$$

$$T(x,y) = BD\sin\left(n\frac{\pi}{10}x\right)e^{-n\frac{\pi}{10}y}$$

$$and T(x,0) = 100 \Rightarrow BD\sin\left(n\frac{\pi}{10}x\right)\cdot 1 = 100$$

Contd…

Solution

$$T(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\frac{\pi}{10}y} \sin\left(n\frac{\pi}{10}x\right)$$

$For\ y = 0,\ we\ must\ have\ T = 100, then:$

$$T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{10}x\right)$$

which is just the Fourier sine series $\cdots$

The Diffusion or Heat Flow Equation:

Example (Flow of Heat through a slab of thickness ℓ)

We have a slab place on the region:
$R : \{0 < x < \ell, \quad -\infty < y < \infty\}$ with border conditions:
$u(x = 0) = 0$ and $u(x = \ell) = 100^\circ$ for $t = 0$ (beginning, a steady-state temperature distribution)

Solution

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}, \quad (\alpha^2 \textit{ is a const. char. mate. and } u \textit{ the temp.})$$

$$\textit{if } u = F(x, y, z)T(t)$$

$$\Rightarrow T\nabla^2 F = \frac{F}{\alpha^2}\frac{\partial T}{\partial t} \quad (\textit{dividing by } u = FT)$$

$$\frac{1}{F}\nabla^2 F = \frac{1}{\alpha^2 T}\frac{\partial T}{\partial t} \Rightarrow \frac{1}{F}\nabla^2 F = -k^2 \quad \textit{and} \quad \frac{1}{\alpha^2 T}\frac{\partial T}{\partial t} = -k^2$$

Solution

The solution of the DFQ dependent of the time is trivial:

$$T(t) = e^{-k^2\alpha^2 t}$$

and in our particular problem, where we have a very long slab, the diffusion will be only on the x-direction, then the DFQ dependent on the space coordinates, also become trivial, because we recognize as the SHO.

$$\frac{\partial^2 F(x)}{\partial x^2} + k^2 F(x) = 0$$

$$soln:$$

$$F(x) = A\cos kx + B\sin kx$$

Solution

B.C.: $u(x=0)=0 \Rightarrow A=0$, *and we allow* $u(x=\ell)=0$ *at a later time(diffusion). Then if* $u(x=\ell)=0 \Rightarrow \sin k\ell = 0 \Rightarrow k\ell = n\pi$
$\Rightarrow k = \frac{n\pi}{\ell}$ *(eigenvalues)*
Then our base functions (eigenfunctions) are then

$$u = e^{-k^2\alpha^2 t}\sin\frac{n\pi x}{\ell}$$

and the general solution to our problem will be the series:

$$u = \sum_{n=1}^{\infty} b_n e^{-k^2\alpha^2 t}\sin\frac{n\pi x}{\ell}$$

Solution

The problem say at the beginning, a steady-state temperature distribution is on the slab, which implies u_0 *satisfy Laplace's equation, i.e* $\nabla^2 u_0 = 0$, *i.e.* $\frac{\partial u_0^2}{\partial x^2} = 0$, *and the solution for this equation is* $u_0 = ax + b$, *then applying border condition we get*

$$u_0 = \frac{100}{\ell} x$$

Then for $t = 0$, $u(x, t = 0) = u_0$, *i.e.*

$$u(x, t = 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} = \frac{100}{\ell} x$$

Which can be solved using F.S., for $f(x) = \frac{100}{\ell} x$ *and the half-period equal to* ℓ

Solution

The Euler coefficients:

$$a_n = 0 \quad (\textit{we dont want solutions with cosine functions (B.C.)})$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \frac{1}{\ell}\left(\int_{-\ell}^{0} 0\,dx + \int_{0}^{\ell}\frac{100}{\ell}x\sin\frac{n\pi x}{\ell}\,dx\right)$$

$$= \frac{100}{\ell^2}\int_0^{\pi} x\sin\frac{n\pi x}{\ell}\,dx \Rightarrow (\textit{Integration by parts:})$$

$$b_n = \frac{100(\sin\pi n - \pi n\cos\pi n)}{\pi^2 n^2} = \frac{100}{\pi}\frac{(-1)^{n-1}}{n}$$

$$\Rightarrow$$

$$u = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-k^2\alpha^2 t}\sin\frac{n\pi x}{\ell}$$

The Wave Equation:

Example (The Vibrating String)

Let a string be stretched tightly and its ends fastened to supports at $x = 0$ and $x = \ell$. When the string is vibrating, its vertical displacement y from its equilibrium position along the x-axis depends on x and t. We assume the displacement y is very small and that the slope $\partial y/\partial t$ is small at any point at any time.

Solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \quad (v = \sqrt{T/\mu})$$

$$\textit{if } y = X(x)T(t)$$

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{v^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -k^2$$

(soln's:)

$$X(x) = A\cos kx + B\sin kx \quad \textit{and} \quad T(t) = C\cos kvt + D\sin kvt$$

Solution

B.C.: $y(x=0)=0 \Rightarrow A=0,$
$u(x=\ell)=0 \Rightarrow \sin k\ell = 0 \Rightarrow k\ell = n\pi \Rightarrow k = \frac{n\pi}{\ell}$ *(eigenvalues)*
Then: $y_1(x,t) = B\sin\frac{n\pi}{\ell}x\left(C\cos\frac{n\pi}{\ell}vt + D\sin\frac{n\pi}{\ell}vt\right)$ *but at* $t=0$ *every piece of string is not varying with time, the* $\partial y/\partial t = 0$
Then $D=0$ *as well.* ☺
Then our base functions (eigenfunctions) are then

$$y_1 = BC\sin\frac{n\pi}{\ell}x\cos\frac{n\pi}{\ell}vt$$

and the general solution to our problem will be the series:

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}vt\right)$$

Solution

Then at $t = 0$, $y(x,t) = y_0 = f(x)$, *then*

$$y(x, t=0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right) = f(x)$$

...

THANK YOU

UNIT-II

PARTIAL DIFFERENTIAL EQUATIONS: FIRST ORDER

Formation of Partial Differential equations

Partial Differential Equation can be formed either by elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables .

SOLVED PROBLEMS

1. Eliminate two arbitrary constants a and b from $(x-a)^2+(y-b)^2+z^2=R^2$ here R is known constant .

(OR) Find the differential equation of all spheres of fixed radius having their centers in x y- plane.

solution

$$(x-a)^2+(y-b)^2+z^2=R^2 \ldots\ldots(1)$$

Differentiating both sides with respect to x and y

$$2z\frac{\partial z}{\partial x}=-2(x-a)$$

$$2z\frac{\partial z}{\partial y}=-2(y-b)$$

$$\frac{\partial z}{\partial x}=p,\frac{\partial z}{\partial y}=q$$

$$\therefore x-a=-pz, y-b=-qz$$

By substituting all these values in (1)

$$p^2 z^2 + q^2 z^2 + z^2 = R^2$$

$$\Rightarrow z^2 = \frac{R^2}{p^2 + q^2 + 1}$$

or

$$z^2 = \frac{R^2}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

2. Find the partial Differential Equation by eliminating arbitrary functions from $z = f(x^2 - y^2)$

SOLUTION

$$z = f(x^2 - y^2)..........(1)$$

$$d.w.r.to.xandy$$

$$\frac{\partial z}{\partial x} = f^{'}(x^2 - y^2) \times 2x......(2)$$

$$\frac{\partial z}{\partial y} = f^{'}(x^2 - y^2) \times -2y......(3)$$

By $\dfrac{(2)}{(3)}$

$$\frac{\left(\dfrac{\partial z}{\partial x}\right)}{\left(\dfrac{\partial z}{\partial y}\right)} = \frac{-x}{y}$$

$$\frac{p}{q} = \frac{-x}{y} \Rightarrow py + qx = 0$$

3.Find Partial Differential Equation by eliminating two arbitrary functions from

$$z = yf(x) + xg(y)$$

SOLUTION

$$z = yf(x) + xg(y)......(1)$$

Differentiating both sides with respect to x and y

$$\frac{\partial z}{\partial x} = yf'(x) + g(y)........(2)$$

$$\frac{\partial z}{\partial y} = f(x) + xg'(y)........(3)$$

Again d . w .r. to x and yin equation (2)and(3)

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

$$x \times (2) + y \times (3) \ldots\ldots to \ldots get$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$$

$$xg(y)+yf(x)+xy(f'(x)+g'(y))$$

$$=z+xy\left(f'+g'\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z+xy\left(\frac{\partial^2 z}{\partial x\partial y}\right)$$

Different Integrals of Partial Differential Equation

1. Complete Integral (solution)

Let $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = F(x, y, z, p, q) = 0 ......(1)$

be the Partial Differential Equation.
The complete integral of equation (1) is given by $\phi(x, y, z, a, b) = 0 ....(2)$

2. Particular solution

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called particular solution .

3.Singular solution

The eliminant of a , b between

$$\phi(x, y, z, a, b) = 0$$

$$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$$

when it exists , is called singular solution

4. General solution

In equation (2) assume an arbitrary relation of the form $b = f(a)$. Then (2) becomes

$$\phi(x, y, z, a, f(a)) = 0 \ldots\ldots\ldots(3)$$

Differentiating (2) with respect to a,

$$\frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b} f'(a) = 0 \ldots\ldots\ldots(4)$$

The eliminant of (3) and (4) if exists, is called general solution

Standard types of first order equations

TYPE-I

The Partial Differential equation of the form

$$f(p,q)=0$$

has solution

$$z=ax+by+c \quad \text{with} \quad f(a,b)=0$$

TYPE-II

The Partial Differential Equation of the form

$z = px+qy+f(p,q)$ is called ***Clairaut's*** form of *pde* , it's solution is given by

$$z=ax+by+f(a,b)$$

TYPE-III

If the *pde* is given by $f(z, p, q) = 0$

then assume that

$$z = \phi(x + ay)$$

$$u = x + ay$$

$$z = \phi(u)$$

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}=\frac{\partial z}{\partial u}.1=\frac{dz}{du}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}=\frac{\partial z}{\partial u}.a=a\frac{dz}{du}$$

$\therefore$ The given *pde* can be written as

$f(z,\frac{dz}{dx},a\frac{dz}{dy})=0$ And also this can be integrated to get solution

TYPE-IV

The *pde* of the form $f(x,p)=g(y,q)$ can be solved by assuming

$$f(x,p)=g(y,q)=a$$

$$f(x,p)=a \Rightarrow p=\phi(x,a)$$

$$g(y,q)=a \Rightarrow q=\Psi(y,a)$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$dz=\phi(x,a)dx+\Psi(y,a)dy$$

Integrate the above equation to get solution

SOLVED PROBLEMS

1.Solve the *pde* $p^2 - q = 1$ and find the complete and singular solutions

Solution

Complete solution is given by

$$z = ax + by + c$$

with $$a^2 - b = 1$$

$$\Rightarrow b = a^2 - 1$$

$$z = ax + (a^2 - 1)y + c$$

d.w.r.to. a and c then

$$\frac{\partial z}{\partial a} = x + 2ay$$

$$\frac{\partial z}{\partial c} = 1 = 0$$ Which is not possible

Hence there is no singular solution

2.Solve the *pde* $pq + p + q = 0$ and find the complete, general and singular solutions

Solution

The complete solution is given by

$$z = ax + by + c$$

with
$$ab + a + b = 0$$

$$a = \frac{-b}{b+1}$$

$$\therefore z = \frac{-b}{b+1}x + by + c.......(1)$$

$$\frac{\partial z}{\partial b} = \frac{-1}{(b+1)^2} x + y = 0$$

$$\frac{\partial z}{\partial c} = 1 = 0 \quad \text{no singular solution}$$

To get general solution assume that

$$c = g(b)$$

From eq (1)

$$\therefore z = \frac{-b}{b+1} x + by + g(b) \ldots\ldots.(2)$$

$$\frac{\partial z}{\partial c} = \frac{-1}{(b+1)^2} x + y + g'(b) \ldots\ldots\ldots (3)$$

Eliminate from (2) and (3) to get general solution

3.Solve the *pde* $z = px + qy + \sqrt{1 + p^2 + q^2}$ and find the complete and singular solutions

Solution

The pde $z = px + qy + \sqrt{1 + p^2 + q^2}$ is in Clairaut's form complete solution of (1) is

$$z = ax + by + \sqrt{1 + a^2 + b^2} \quad \ldots\ldots.(2)$$

d.w.r.to "a" and "b"

$$\left.\begin{aligned} \frac{\partial z}{\partial a} &= x + \frac{a}{\sqrt{1 + a^2 + b^2}} = 0 \\ \frac{\partial z}{\partial b} &= y + \frac{b}{\sqrt{1 + a^2 + b^2}} = 0 \end{aligned}\right\} \quad \ldots\ldots..(3)$$

From (3)

$$x^2 = \frac{a^2}{1+a^2+b^2}, y^2 = \frac{b^2}{1+a^2+b^2}$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$\Rightarrow \frac{1}{1+a^2+b^2} = 1-(x^2+y^2)$$

$$ax+\frac{a^2}{\sqrt{1+a^2+b^2}}=0$$

$$by+\frac{b^2}{\sqrt{1+a^2+b^2}}=0$$

$$ax+by+\sqrt{1+a^2+b^2}-\frac{1}{\sqrt{1+a^2+b^2}}=0$$

$$z-\frac{1}{\sqrt{1+a^2+b^2}}=0 \Rightarrow z^2=1-(x^2+y^2)$$

$\Rightarrow x^2+y^2+z^2=1$ is required singular solution

4.Solve the pde $(1-x)p+(2-y)q=3-z$

Solution

pde $$(1-x)p+(2-y)q=3-z$$

$$z = px+qy+(3-p-2q)$$

Complete solution of above pde is

$$z = ax+by+(3-a-2b)$$

5.Solve the pde $p^2+q^2=z$

Solution

Assume that $$z=\phi(x+ay)$$

$$u = x + ay$$

$$z = \phi(u)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}.1 = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}.a = a\frac{dz}{du}$$

From given pde

$$p^2 + q^2 = z \Rightarrow \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z}{1+a^2}$$

$$\left(\frac{dz}{du}\right) = \sqrt{\frac{z}{1+a^2}} \Rightarrow \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}du$$

Integrating on both sides

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + b$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + b$$

6. Solve the pde $zpq = p + q$

Solution

Assume $q = ap$

Substituting in given equation

$$zpap = p + ap$$

$$p = \frac{1+a}{az}, q = \frac{1+a}{z}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = \frac{1+a}{az} dx + \frac{1+a}{z} dy$$

$$zadz = (1+a)(dx + ady)$$

Integrating on both sides

$$\frac{a}{2}z^2 = (1+a)(x+ay) + b$$

7.Solve pde $pq = xy$

(or) $\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = xy$

Solution

$$\frac{p}{x} = \frac{q}{y}$$

Assume that

$$\frac{p}{x} = \frac{y}{q} = a$$

$$\therefore\ p = ax, q = \frac{y}{a}$$

$$dz = \ pdx + qdy = axdx + \frac{y}{a}dy$$

Integrating on both sides

$$z = a\frac{x^2}{2} + \frac{y^2}{2a} + b$$

8. Solve the equation $p^2+q^2=x+y$

Solution

$$p^2-x=y-q^2=a$$

$$p=\sqrt{a+x},q=\sqrt{y-a}$$

$$dz=pdx+qdy=\sqrt{a+x}dx+\sqrt{y-a}dy$$

integrating

$$z=\frac{2}{3}(a+x)^{\frac{3}{2}}+(y-a)^{\frac{3}{2}}+b$$

Equations reducible to the standard forms

(i)If $(x^m p)$ and $(y^n q)$ occur in the ***pde*** as in

$F(x^m p, y^n q) = 0$ Or in $F(z, x^m p, y^n q) = 0$

Case (a) Put $x^{1-m} = X$ and $y^{1-n} = Y$
if $m \neq 1$; $n \neq 1$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)x^{-m}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial Y}(1-n)y^{-n}$$

$$x^m p = \frac{\partial z}{\partial X}(1-m) = P(1-m)$$

$$y^n q = \frac{\partial z}{\partial Y}(1-n) = Q(1-n)$$

where $\dfrac{\partial z}{\partial X} = P, \dfrac{\partial z}{\partial Y} = Q$

Then $F(x^m p, y^n q) = 0$ reduces to $F(P,Q) = 0$

Similarly $F(z, x^m p, y^n q) = 0$ reduces to $F(z, P, Q) = 0$

case(b)

If $m=1$ or $n=1$

put $\log x = X, \log y = Y$

$$p = \frac{\partial z}{\partial X}\frac{1}{x} \Longrightarrow px = P$$

$$q = \frac{\partial z}{\partial Y}\frac{1}{y} \Longrightarrow qy = Q$$

(ii)If $(z^k p)$ and $(z^k q)$ occur in ***pde*** as in $F(z^k p, z^k q)$

Or in $f_1(x, z^k p) = f_2(y, z^k q)$

Case(a) Put $z^{1+k} = Z$ if $k \neq -1$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = z^{-k}(1+k)^{-1}\frac{\partial Z}{\partial x} \Rightarrow z^{k}p = (1+k)^{-1}P$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y} = z^{-k}(1+k)^{-1}\frac{\partial Z}{\partial y} \Rightarrow z^{k}q = (1+k)^{-1}Q$$

where $\frac{\partial Z}{\partial x} = P, \frac{\partial Z}{\partial y} = Q$

Given pde reduces to

$F(P,Q)$ and $f_1(x,P) = f_2(y,Q)$

Case(b) if $k = -1$

$$\log z = Z$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \Rightarrow z^{-1}p = P$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y} = z\frac{\partial Z}{\partial y} \Rightarrow z^{-1}q = Q$$

Solved Problems

1.Solve $p^2x^4 + q^2y^4 = z^2$

Solution $\left(\frac{px^2}{z}\right)^2 + \left(\frac{qy^2}{z}\right)^2 = 1.......(1)$

$$m=2, n=2$$

$$k=-1$$

$$x^{-1}=X \qquad y^{-1}=Y \qquad \log z=Z$$

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial Z}\frac{\partial Z}{\partial X}\frac{\partial X}{\partial x}=-zx^{-2}\frac{\partial Z}{\partial X}=-zx^{-2}P$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Z}\frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y}=-zy^{-2}\frac{\partial Z}{\partial Y}=-zy^{-2}Q$$

where $\frac{\partial Z}{\partial X}=P, \frac{\partial Z}{\partial Y}=Q$

$$\frac{px^2}{z} = -P, \frac{qy^2}{z} = -Q$$

(1)becomes

$$(-P)^2 + (-Q)^2 = 1$$

$$P^2 + Q^2 = 1$$

$$\therefore Z = aX + bY + c$$

$$a^2 + b^2 = 1, b = \sqrt{1-a^2}$$

$$\log z = ax^2 + \sqrt{1-a^2}\, y^2 + c$$

2. Solve the pde $p^2+q^2=z^2(x^2+y^2)$

SOLUTION

$$\left(\frac{p}{z}\right)^2+\left(\frac{q}{z}\right)^2=(x^2+y^2)\ldots\ldots(1)$$

$$k=-1 \quad \log z=Z$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}=z\frac{\partial Z}{\partial x}\Rightarrow z^{-1}p=P$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y}=z\frac{\partial Z}{\partial y}\Rightarrow z^{-1}q=Q$$

Eq(1) becomes

$$P^2+Q^2=(x^2+y^2).....(2)$$

$$P^2-x^2=y^2-Q^2=a^2$$

$$\log z=\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}(a^2+x^2)+$$

$$\frac{y\sqrt{(y^2-a^2)}}{2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)+b$$

Lagrange's Linear Equation

Def: The linear partial differenfial equation of first order is called as Lagrange's linear Equation.

This eq is of the form $Pp + Qq = R$

Where P, Q and R are functions x,y and z

The general solution of the partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$

Where F is arbitrary function of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

Here $u = c_1$ and $v = c_2$ are independent solutions of the auxilary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Solved problems

1.Find the general solution of $x^2 p + y^2 q = (x+y)z$

Solution

auxilary equations are $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{(x+y)z}$

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$ Integrating on both sides

$$u = \left(x^{-1} - y^{-1}\right) = c_1$$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{(x-y)(x+y)} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{(x-y)} = \frac{dz}{z}$$ Integrating on both sides

$$\log(x-y)=\log z+\log c_2$$

$$v=(x-y)z^{-1}=c_2$$

The general solution is given by $F(u,v)=0$

$$F(x^{-1}-y^{-1},(x-y)z^{-1})=0$$

2.solve $x^2(y-z)+y^2(z-x)q=z^2(x-y)$

solution

Auxiliary equations are given by

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

$$\frac{\frac{dx}{x^2}}{(y-z)} = \frac{\frac{dy}{y^2}}{(z-x)} = \frac{\frac{dz}{z^2}}{(x-y)}$$

$$\frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{(y-z)+(z-x)+(x-y)}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating on both sides

$$u = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$$

$$\frac{x^{-1}dx}{x(y-z)} = \frac{y^{-1}dy}{y(z-x)} = \frac{z^{-1}dz}{z(x-y)}$$

$$\frac{x^{-1}dx + y^{-1}dy + z^{-1}dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$ Integrating on both sides

$$v = xyz = b$$

The general solution is given by

$$F(x^{-1}+y^{-1}+z^{-1}, xyz)=0$$

HOMOGENEOUS LINEAR PDE WITH CONSTANT COEFFICIENTS

Equations in which partial derivatives occurring are all of same order (with degree one) and the coefficients are constants ,such equations are called homogeneous linear PDE with constant coefficient

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots\dots\dots a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Assume that $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$.

then n^{th} order linear homogeneous equation is given by

$$(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots\dots\dots + a_n D'^n) z = F(x, y)$$

or

$$f(D, D') z = F(x, y) \dots\dots\dots (1)$$

The complete solution of equation (1) consists of two parts ,the complementary function and particular integral.

The complementary function is complete solution of equation of $f(D,D')z=0$

Rules to find complementary function

Consider the equation

$$\frac{\partial^2 z}{\partial x^2}+k_1\frac{\partial^2 z}{\partial x\partial y}+k_2\frac{\partial^2 z}{\partial y^2}=0$$

or

$$(D^2+k_1DD'+k_2D'^2)z=0............(2)$$

The auxiliary equation for (A.E) is given by

$$D^2 + k_1 DD' + k_2 D'^2 = 0$$

And by giving $D = m, D' = 1$

The A.E becomes $m^2 + k_1 m + k_2 = 0....(3)$

Case 1

If the equation(3) has two distinct roots m_1, m_2

The complete solution of (2) is given by

$$z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

Case 2

If the equation(3) has two equal roots i.e $m_1 = m_2$

The complete solution of (2) is given by

$$z = f_1(y + m_1 x) + x f_2(y + m_1 x)$$

Rules to find the particular Integral

Consider the equation

$$(D^2 + k_1 DD' + k_2 D'^2)z = F(x, y)$$

$$f(D, D')z = F(x, y)$$

Particular Integral (P.I) $= \dfrac{F(x,y)}{f(D,D')}$

Case 1 If $F(x,y) = e^{ax+by}$

then P.I= $\dfrac{1}{f(D,D')} e^{ax+by}$

$= \dfrac{1}{f(a,b)} e^{ax+by}, \, f(a,b) \neq 0$

If $f(a,b)=0$ and $(D - \frac{a}{b} D')$ is

factor of $f(D,D')$ then

P.I $= xe^{ax+by}$

If $f(a,b)=0$ and $\left(D-\frac{a}{b}D'\right)^2$ is factor of $f(D,D')$

then P.I $= \frac{x^2}{2}e^{ax+by}$

Case 2

$$F(x,y)=\sin(mx+ny)\,or\cos(mx+ny)$$

P.I $= \dfrac{\sin(mx+ny)}{f(D^2,DD',D'^2)} = \dfrac{\sin(mx+ny)}{f(-m^2,-mn,-n^2)}$

Case 3 $F(x,y)=x^m y^n$

P.I $=\dfrac{1}{f(D,D')}x^m y^n=\left[f(D,D')\right]^{-1}x^m y^n$

Expand $\left[f(D,D')\right]^{-1}$ in ascending powers of D or D' and operating on $x^m y^n$ term by term.

Case 4 when $F(x,y)$ is any function of x and y.

P.I= $\dfrac{1}{f(D,D')}F(x,y)$

$$\frac{1}{D-mD'}F(x,y)=\int F(x,c-mx)dx$$

Here $(D-mD')$ is factor of $f(D,D')$

Where 'c' is replaced by $(y+mx)$ after integration

Solved problems

1.Find the solution of ***pde***

$$(D^3-D'^3+3DD'^2-3D^2D')z=0$$

Solution

The Auxiliary equation is given by

Solution

The Auxiliary equation is given by

$$m^3 - 1 + 3m - 3m^2 = 0$$

By taking $D = m, D' = 1$

$$\therefore m = 1,1,1.$$

Complete solution $= f_1(y+x) + xf_2(y+x) + x^2 f_3(y+x)$

2. Solve the ***pde*** $(D^3 + 4D^2D' - 5DD')z = 0$

Solution

The Auxiliary equation is given by

$$m^3 + 4m^2 - 5m = 0$$

$$\therefore m = 0, 1, -5$$

$$z = f_1(y) + f_2(y + x) + f_3(y - 5x)$$

3. Solve the ***pde*** $(D^2 + D'^2)z = 0$

Solution

the A.E is given by $m^2 + 1 = 0$

$$m = \pm i$$

$$\therefore z = f_1(y + ix) + f_2(y - ix)$$

4. Find the solution of ***pde***

$$(D^2+3DD'-4D'^2)z=e^{2x+4y}$$

Solution

Complete solution =
Complementary Function + Particular Integral

The A.E is given by $m^2+3m-4=0$

$$m=-4,1$$

$$C.F=\phi_1(y+x)+\phi_2(y-4x)$$

$$P.I = \frac{e^{2x+4y}}{D^2 + 3DD' - 4D'^2} = \frac{e^{2x+4y}}{-36}$$

Complete solution

$$= C.F + P.I$$

$$= \phi_1(y+x) + \phi_2(y-4x) - \frac{e^{2x+4y}}{36}$$

5.Solve $(D^3 - 3DD' + 2D'^3)z = e^{2x-y} + e^{x+y}$

Solution

$$A.E = m^3 - 3m + 2$$

$$\therefore m = 1, 1, -2.$$

$$C.F = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x)$$

$$P.I_1 = \frac{e^{2x-y}}{D^3 - 3DD'^2 + 2D'^3} = \frac{e^{2x-y}}{(D-D')^2(D^2+2D')}$$

$$P.I_1 = \frac{e^{2x-y}}{(D-D')^2(D+2D')} = \frac{xe^{2x-y}}{9}$$

$$P.I_2 = \frac{e^{x+y}}{D^3 - 3DD'^2 + 2D'^3} = \frac{e^{x+y}}{(D-D')^2(D^2+2D')}$$

$$\therefore P.I_2 = \frac{x^2}{6} e^{x+y}$$

$$z = C.F + P.I_1 + P.I_2$$

$$z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-2x)+\frac{xe^{2x-y}}{9}+\frac{x^2}{6}e^{x+y}$$

6.Solve $(D^2-DD')z=\cos x\cos 2y$

Solution

$$(D^2-DD')z=\frac{1}{2}\left[\cos(x+2y)+\cos(x-2y)\right]$$

$$A.E=m^2-m=0$$

$$m=0,1$$

$$C.F=\phi_1(y+x)+\phi_2(y)$$

$$P.I_1 = \frac{\cos(x+2y)}{(D^2 - DD')} = \frac{\cos(x+2y)}{((-1)-(-2))} = \cos(x+2y)$$

$$P.I_2 = \frac{\cos(x-2y)}{(D^2 - DD')} = \frac{\cos(x-2y)}{((-1)-(2))} = \frac{\cos(x-2y)}{-3}$$

$$z = \phi_1(y+x) + \phi_2(y-x) + \cos(x+2y) - \frac{1}{3}\cos(x-2y)$$

7.Solve $(D^2 + DD' - 6D'^2)z = x^2y^2$

Solution $A.E = m^2 + m - 6 = 0$

$m = 2, -3.$

$$C.F = \phi_1(y+2x) + \phi_2(y-3x)$$

$$P.I = \frac{x^2 y^2}{D^2 + DD' - 6D'^2}$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right]^{-1} x^2 y^2$$

$$= D^{-2}\left[1 - \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right) + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)^2\right] x^2 y^2$$

$$= D^{-2}\left[1-\left(\frac{D'}{D}-6\frac{D'^2}{D^2}\right)+\frac{D'^2}{D^2}\right]x^2y^2$$

$$= D^{-2}\left[x^2y^2-\left(\frac{2x^2y}{D}-6\frac{2x^2}{D^2}\right)+\frac{2x^2}{D^2}\right]$$

$$= D^{-2}\left[x^2y^2-\left(\frac{2x^3y}{3}-6\frac{2x^4}{12}\right)+\frac{2x^4}{12}\right]$$

$$= D^{-2}\left[x^2y^2+\frac{2x^3y}{3}+8\frac{2x^4}{12}\right]$$

$$=\left[\frac{x^4y^2}{12}+\frac{2x^5y}{60}+\frac{2x^6}{90}\right]$$

7.Solve $(D^2 - 5DD' + 6D'^2)z = y\sin x$

Solution

A.E is $m^2 - 5m + 6 = 0$

$$m = 3, m = 2.$$

$$C.F = \phi_1(y + 3x) + \phi_2(y + 2x)$$

$$P.I = \frac{y\sin x}{D^2 - 5DD' + 6D'^2} = \frac{y\sin x}{(D - 3D')(D - 2D')}$$

$$= \frac{1}{(D - 3D')}\left[\frac{y\sin x}{(D - 2D')}\right]$$

$$= \frac{1}{(D - 3D')}\int (a - 2x)\sin x dx$$

$$= \frac{1}{(D - 3D')}[-a\cos x - 2(-x\cos x + \sin x)]$$

$$= \frac{1}{(D - 3D')}[2x\cos x - 2\sin x - (y + 2x)\cos x]$$

$$P.I = \frac{y\sin x}{D^2 - 5DD' + 6D'^2} = \frac{y\sin x}{(D-3D')(D-2D')}$$

$$= \frac{1}{(D-3D')}\left[\frac{y\sin x}{(D-2D')}\right]$$

$$= \frac{1}{(D-3D')}\int (a-2x)\sin x\,dx$$

here

$$(a = y + 2x)$$

$$= \frac{1}{(D-3D')}\left[-a\cos x - 2(-x\cos x + \sin x)\right]$$

$$= \frac{1}{(D-3D')}\left[2x\cos x - 2\sin x - (y+2x)\cos x\right]$$

$$= \frac{1}{(D-3D')}\left[-y\cos x - 2\sin x\right]$$

$$= \int(-(b-3x)\cos x - 2\sin x)dx$$

here

$(b = y + 3x)$

$$= -b\sin x + 2\cos x + 3(x\sin x + \cos x)$$

$$= -(y+3x)\sin x + 2\cos x + 3(x\sin x + \cos x)$$

$$= 5\cos x - y\sin x$$

Non Homogeneous Linear PDES

If in the equation $f(D,D')z = F(x,y)$............(1)

the polynomial expression $f(D,D')$ is not homogeneous, then (1) is a non- homogeneous linear partial differential equation

Ex $(D^2 + 3D + D' - 4D'^2)z = e^{2x+3y}$

Complete Solution

= Complementary Function + Particular Integral

To find C.F., factorize $f(D,D')$

into factors of the form $(D - mD' - c)$

If the non homogeneous equation is of the form

$$(D - m_1 D' - c_1)(D - m_2 D' - c_2)z = F(x, y)$$

$$C.F = e^{c_1 x}\phi(y + m_1 x) + e^{c_2 x}\phi(y + m_2 x)$$

1.Solve $(D^2 - DD' + D)z = x^2$

Solution

$$f(D, D') = D^2 - DD' + D = D(D - D' + 1)$$

$$C.F = e^{-x}\phi_1(y + x) + \phi_2(y)$$

$$P.I = \frac{x^2}{D^2 - DD' + D} = \frac{1}{D^2}\left[1 - \frac{(D'+1)}{D}\right]^{-1} x^2$$

$$= \frac{1}{D^2}\left[x^2 + \left[\frac{(D'+1)}{D}\right]x^2 + \left[\frac{(D'+1)}{D}\right]^2 x^2 + \ldots\ldots\right]$$

$$= \frac{1}{D^2}\left[x^2 + \frac{x^3}{3} + \frac{x^4}{12}\right] = \left[\frac{x^4}{3.4} + \frac{x^5}{3.4.5} + \frac{x^6}{12.5.6}\right]$$

2.Solve $(D+D'-1)(D+2D'-3)z=4$

Solution

$$z=e^{x}\phi_1(y-x)+e^{3x}\phi_1(y-2x)+\frac{4}{3}$$

THANK YOU

Ordinary Differential Equations

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib

Syllabus

The syllabus contains the following articles:

- First Order Differential Equation
  - Leibnitz linear equation
  - Bernoulli's equation
  - Exact differential equation
  - Equations not of first degree
    - Equation solvable for p
    - Equation solvable for x
    - Equation solvable for y
  - Clairaut's equation
- Higher Order Differential Equation
  - Second order linear differential equations with variable coefficients
  - Method of variation of parameters
  - Power series solutions

Leibnitz linear equation

Definition

An equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are either constants or functions of x only is called Leibnitz linear equation.
Alternately, the equation may be of the form $\frac{dx}{dy} + Px = Q$, where P and Q are either constants or functions of y only.

Solution

This equation is solved by evaluating the Integration Factor that is given by $IF = e^{\int Pdx}$ and the solution is obtained by $y(IF) = \int Q(IF)dx + c$ for the former case and for the latter x is replaced by y in the IF and the solution.

Questions

- $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$
- $x \log x \dfrac{dy}{dx} + y = 2 \log x$
- $\dfrac{dy}{dx} + y \cot x = 5e^{\cos x}$
- $\dfrac{dy}{dx} = \dfrac{y}{2y \log y + y - x}$
- $\sqrt{1 - y^2}dx = (\sin^{-1} y - x)dy$

Bernoulli's Equation

Definition

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are either constants or functions of x only is called Bernoulli's equation.
Alternately, the equation may also be written as $\frac{dx}{dy} + Px = Qx^n$, where P and Q are either constants or functions of y only.

Solution

This equation is reduced to Leibnitz linear equation by substituting $y^{1-n} = z$ and differentiating. This generates the Leibnitz equation in z and x that is solved as explained earlier and then z is resubstituted in terms of y. The corresponding changes are made in the latter case of definition.

QUESTIONS

- $x\dfrac{dy}{dx} + y = x^3y^6$
- $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$
- $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$
- $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$
- $(xy^2 - e^{1/x^3})dx - x^2ydy = 0$

Exact Differential Equation

Definition

An equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be an Exact differntial equation if it can be obtained directly by differentiating the equation $u(x,y) = c$, which is its primitive.
i.e. if

$$du = Mdx + Ndy$$

Necessary and Sufficient Condition

The necessary and sufficient condition for the equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

The solution of $Mdx + Ndy = 0$ is given by

$$\int_{y \text{ constant}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$$

Questions

- $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$
- $(1 + e^{x/y})dx + \left(1 - \dfrac{x}{y}\right) e^{x/y} dy = 0$
- $(2xy \cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$
- $xdy + ydx + \dfrac{xdy - ydx}{x^2 + y^2} = 0$
- $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

Equations reducible to exact equations

Reducible to exact equations

Equations which are not exact can sometimes be made exact after multiplying by a suitable factor (function of x and/or y) called the Integration Factor (IF).

IF by Inspection

- $ydx + xdy = d(xy)$
- $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$
- $\dfrac{xdy - ydx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
- $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right]$
- $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
- $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$
- $\dfrac{ydx + xdy}{xy} = d[\log(xy)]$
- $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

Equations reducible to exact equations

IF for Homoegeneous Equation

If $Mdx + Ndy = 0$ is a Homoegeneous equation in x and y, then $\dfrac{1}{Mx + Ny}$ is an IF provided $Mx + Ny \neq 0$.

IF for $f_1(xy)ydx + f_2(xy)xdy = 0$

For equation of this type, IF is given by $\dfrac{1}{Mx - Ny}$.

Equations reducible to exact equations

IF for $Mdx + Ndy = 0$

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $IF = e^{\int f(x)dx}$.
- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only, say $g(y)$, then $IF = e^{\int g(y)dy}$.

IF for $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$

In this equation, a, b, c, d, m, n, p, q are all constants and IF is given by $x^h y^k$, where h and k are so chosen that the equation becomes exact after multiplication with IF.

Questions

- $(1+xy)ydx + (1-xy)xdy = 0$
- $xdy - ydx = xy^2dx$
- $(xye^{x/y} + y^2)dx - x^2e^{x/y}dy = 0$
- $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$
- $\left(y + \dfrac{y^3}{3} + \dfrac{x^2}{2}\right)dx + \dfrac{1}{4}(x + xy^2)dy = 0$
- $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$
- $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

Equations of first order and higher degree

Definition

A differential equation of the first order and n^{th} degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0, \text{ where } p = \frac{dy}{dx} \tag{1}$$

Equations solvable for p

Resolve equation (1) into n linear factors and solve each of the factors to obtain solution of the given equation.

Questions

- $p^2 - 7p + 12 = 0$
- $xyp^2 - (x^2 + y^2)p + xy = 0$
- $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$
- $p^2 - 2p \sinh x - 1 = 0$
- $4y^2p^2 + 2pxy(3x + 1)3x^3 = 0$

Equations solvable for y

Differentiate equation (1), wrt x, to obtain a differential equation of first order in p and x that has solution of the form $\phi(x, p, c) = 0$. The elimination p from this solution and equation (1) gives the desired solution.

Questions

- $xp^2 - 2yp + ax = 0$
- $y - 2px = \tan^{-1}(xp^2)$
- $x^2 \left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$
- $x - yp = ap^2$

Equations solvable for x

Differentiate equation (1), wrt y, to obtain a differential equation of first order in p and y that has solution of the form $\phi(y, p, c) = 0$. The elimination p from this solution and equation (1) gives the desired solution.

Questions

- $y = 3px + 6p^2y^2$
- $p^3 - 4xyp + 8y^2 = 0$
- $y = 2px + p^2y$
- $y^2 \log y = xyp + p^2$

Clairaut's equation

Definition

An equation of the form $y = px + f(p)$ is called Clairaut's equation.

Solution

Differente the equation wrt x, and obtain the solution by putting $p = c$ in the given equation.

Questions

- $y = xp + \dfrac{a}{p}$
- $y = px + \sqrt{a^2p^2 + b^2}$
- $p = \sin(y - px)$
- $p = \log(px - y)$

Linear Differential Equations

Definition

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \tag{2}$$

Linear Differential Equations

Complementary Function (CF)

- If all the roots of equation (2) are real and distint, CF is given by $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
- If two roots are equal, say $m_1 = m_2$, then CF is given by $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
- If two roots are imaginary, say $m_1 = \alpha + \iota\beta,\ m_2 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
- It two pairs of imaginary roots are equal, say $m_1 = m_2 = \alpha + \iota\beta,\ m_3 = m_4 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

Linear Differential Equations

Particular Integral (PI)

- If $X = e^{ax}$, then PI is given by $y = \dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$.
- If $X = \sin(ax+b)$ or $\cos(ax+b)$, then PI is given by $y = \dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)$. Likewise for $\cos(ax+b)$.
- If $X = x^m$, where m is a positive integer, then PI is given by $y = \dfrac{1}{(D)}x^m$.

  Take out the lowest degree term from $f(D)$ to make the first term unity and then shift the remaining term to numerator and apply Binomial expansion upto D^m. Operate term by term on x^m.
- If $X = e^{ax}V$, where V is a function of x, then PI is given by $y = \dfrac{1}{f(D)}e^{ax}V = e^{ax}\dfrac{1}{f(D+a)}V$.
- If X is any other function of x, then PI is obtained by resolving the $f(D)$ into linear factors and applying $\dfrac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx$

QUESTIONS

- $(D^2 + 4D + 5)y = -2\cosh x$
- $(D2 - 4D + 3)y = \sin 3x \cos 2x$
- $(D^2 + 4)y = e^x + \sin 2x$
- $(D^2 + D)y = x^2 + 2x + 4$
- $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$
- $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$
- $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$
- $(D - 1)^2(D + 1)^2 y = \sin^2 \dfrac{x}{2} + e^x + x$

Cauchy's Homogeneous Equation

Definition

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \tag{3}$$

where a_is are constants and X is a function of x is called Cauchy's Homegeneous Linear Equation.

Solution

The equation is reduced to an LDE with constant coefficients by putting $z = e^x$ thereby generating an LDE in x and z that can be solved as explained earlier and finally the solution of equation (3) is obtained by putting $z = \log x$.

Questions

- $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} - 25y = 50$
- $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$
- $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$
- $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x\dfrac{\sin(\log x)+1}{x}$

Legendre's Linear Equation

Definition

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X \quad (4)$$

where a_is, a and b are constants and X is a function of x is called Legendre's Linear Equation.

Solution

The equation is reduced to an LDE with constant coefficients by putting $a+bx = e^z$ thereby generating an LDE in x and z that can be solved as explained earlier and finally the solution of equation (4) is obtained by putting $z = \log(a+bx)$.

QUESTIONS

- $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos\log(1+x)$
- $(1+2x)^2\dfrac{d^2y}{dx^2}-6(1+2x)\dfrac{dy}{dx}+16y=8(1+2x)^2$
- $(3+2x)^2\dfrac{d^2y}{dx^2}-2(3+2x)\dfrac{dy}{dx}-12y=6x$

Variation of Parameters

This method is applicable for the second order differential equation of the form $\dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_2y = X$
Let the CF of this equation be

$$y = c_1y_1 + c_2y_2$$

. Then the PI of this equation is given by

$$y = uy_1 + vy_2$$

where

$$u = -\int \frac{y_2X}{W}dx$$

and

$$v = \int \frac{y_1X}{W}dx$$

where W is the Wronskian of $y_1,\ y_2$.

QUESTIONS

- $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$
- $\dfrac{d^2y}{dx^2} + y = cosec\ x$
- $\dfrac{d^2y}{dx^2} + y = x\sin x$
- $y'' - 2y' + 2y = e^x \tan x$

Series Solution

We discuss the method of solving equations of the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{5}$$

where $P_0(x), P_1(x)$ and $P_2(x)$ are polynomials in x, in terms of infinite convergent series.

Solution

Divide equation (5) by $P_0(x)$ to get

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{6}$$

where $p(x) = \dfrac{P_1(x)}{P_0(x)}$ and $q(x) = \dfrac{P_2(x)}{P_0(x)}$

Series Solution

Ordinary Point

$x = 0$ is called an ordinary point of equation (5) if $P_0(0) \neq 0$.
In this casem the solution of equation (5), can be expressed as

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Singular Point

$x = 0$ is called a singular point of equation (5), if $P_0(0) = 0$.
In this case, the solution of equation (5) can be expressed as

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Solution when $x = 0$ is an ordinary point

Solution

Let $y = \sum_{k=0}^{\infty} a_k x^k$ be the solution of equation (5). Then, on differentiating

$\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$ and $\frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$.

1. Substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (5).
2. Equate to zero the coefficients of various powers of x and find $a_2, a_3, a_4, \ldots$ in terms of a_0 and a_1.
3. Equate to zero the coefficient of x^n. The relation so obtained is called the recurrence relation.
4. Give different values to n in the recurrence relation to determine various a_is in terms of a_0 and a_1.
5. Substitute the values in the above mentioned series to obtain the solution with a_0 and a_1 as arbitrary constants.

QUESTIONS

- $\dfrac{d^2y}{dx^2} + xy = 0$
- $y'' - xy' + x^2y = 0$
- $(2 - x^2)y'' + 2xy' - 2y = 0$

SOLUTION WHEN $x = 0$ IS A REGULAR SINGULAR POINT I

Let $y = \sum_{k=0}^{\infty} a_k x^{m+k}$ be the solution of equation (5). Then, on differentiating

$$\frac{dy}{dx} = \sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}.$$

1. Substitute the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (5).
2. Equate to zero the coefficients of lowest powers of x. This gives a quadratic equation in m, which in known as indicial equation.
3. Equate to zero the coefficients of other powers of x to find $a_1, a_2, a_3, a_4, \ldots$ in terms of a_0.
4. Substitute the values of $a_1, a_2, a_3, \ldots$ in above said solutionto get the series solution of (5) having a_0 as the arbitrary constant. Though, it is not the complete solution as the same should have two arbitrary constants.
5. The method of complete solution depends on the nature of roots of the indicial equation.

Solution when $x = 0$ is a regular singular point II

CASE I When the roots m_1, m_2 are distinct and not differing by an integer. Then the complete solution is given by

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

CASE II When the roots m_1, m_2 are equal. Then the complete solution is given by

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

CASE III When the roots $m_1 < M_2$ are distinct and differ by an integer. Then th ecomplete solution is given by

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

QUESTIONS

- $2x^2\dfrac{d^2y}{dx^2}+(2x^2-x)\dfrac{dy}{dx}+y=0$
- $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+(x^2-4)y=0$
- $2x(1-x)\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}+3y=0$