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**Subject:** Mathematics-III **Code:** BTAM301-18

**Semester:** 3rd **Branch:** Civil Engg.

# Unit-I

## Vector Calculus – I

# Directional Derivatives and Gradients

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# Directional Derivative

# Directional Derivative

You are standing on the hillside represented by $z = f(x)$ in Figure 13.42 and want to determine the hill's incline toward the $z$- axis.

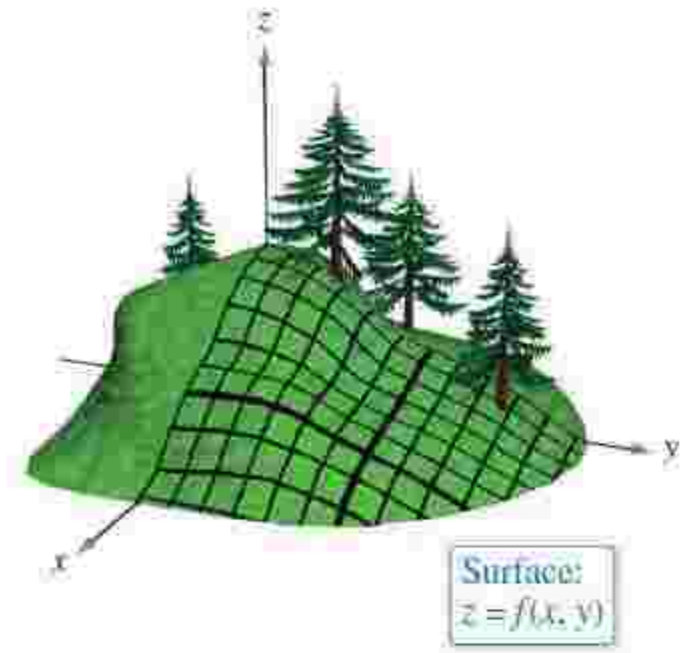



**Figure 13.42**

# Directional Derivative

You already know how to determine the slopes in two different directions—the slope in the *y*-direction would be given by the partial derivative $f_y(x, y)$, and the slope in the *x*-direction would be given by the partial derivative $f_x(x, y)$.

In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

# Directional Derivative

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.**

Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of $f$, as shown in Figure 13.43.

The "direction" of the directional derivative is given by a unit vector

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

where $\theta$ is the angle the vector makes with the positive $x$-axis.

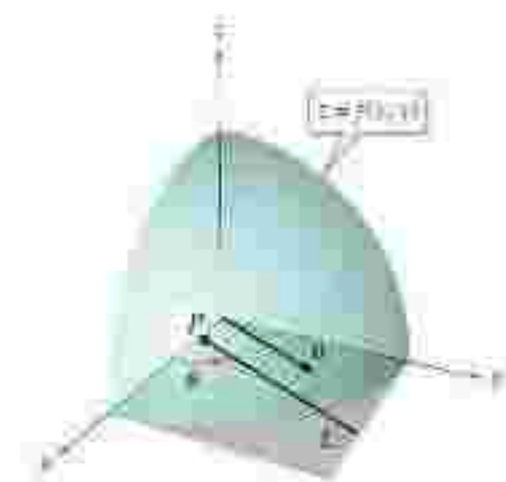


**Figure 13.43**

# Directional Derivative

To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point *P* and parallel to **u**, as shown in Figure 13.44.

This vertical plane intersects the surface to form a curve *C*.

The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of **u** is defined as the slope of the curve *C* at that point.

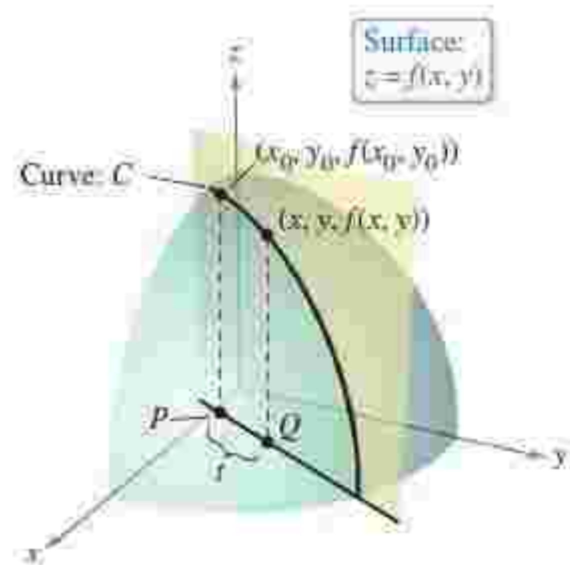



Figure 13.44

# Directional Derivative

Informally, you can write the slope of the curve *C* as a limit that looks much like those used in single-variable calculus.

The vertical plane used to form *C* intersects the *xy*-plane in a line *L*, represented by the parametric equations

$$x = x_0 + t\cos\theta \qquad \text{and} \qquad y = y_0 + t\sin\theta$$

so that for any value of *t*, the point $Q(x, y)$ lies on the line *L*.

For each of the points *P* and *Q*, there is a corresponding point on the surface.

$(x_0, y_0, f(x_0, y_0))$ Point above *P*

$(x, y, f(x, y))$ Point above Q

# Directional Derivative

Moreover, because the distance between *P* and *Q* is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2}$$
$$= |t|$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Finally, by letting *t* approach 0, you arrive at the following definition.

# Directional Derivative

**Definition of Directional Derivative**

Let $f$ be a function of two variables $x$ and $y$ and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the **directional derivative of** $f$ **in the direction of u**, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

Calculating directional derivatives by this definition is similar to finding the derivative of one variable by the limiting process. A simpler “working” formula for finding directional derivatives involves the partial derivatives $f_x$ and $f_y$.

# Directional Derivative

**THEOREM 13.9 Directional Derivative**

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

# Directional Derivative

There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by **u**, as shown in Figure 13.45.

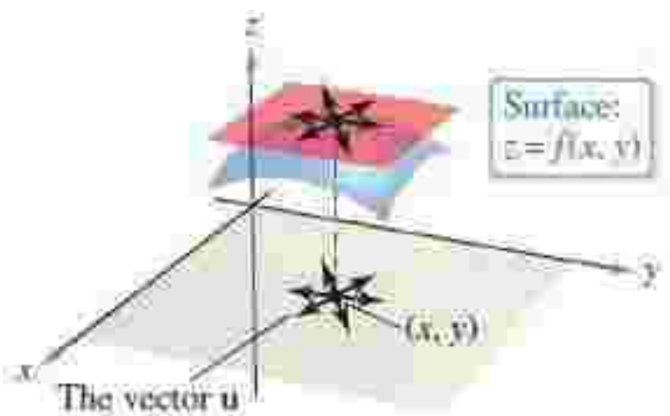



**Figure 13.45**

# Directional Derivative

Two of these are the partial derivatives $f_x$ and $f_y$.

**1.** Direction of positive $x$-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y)$$

**2.** Direction of positive $y$-axis ($\theta = \pi/2$): $\mathbf{u} = \cos\dfrac{\pi}{2}\,\mathbf{i} + \sin\dfrac{\pi}{2}\,\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y)$$

# Example 1 – *Finding a Directional Derivative*

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \frac{1}{4}y^2 \qquad \text{Surface}$$

at (1, 2) in the direction of

$$\mathbf{u} = \left(\cos \frac{\pi}{3}\right)\mathbf{i} + \left(\sin \frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

# Example 1 – *Solution*

Because $f_x$ and $f_y$ are continuous, $f$ is differentiable, and you can apply Theorem 13.9.

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

$$= (-2x)\cos\theta + \left(-\frac{y}{2}\right)\sin\theta$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$D_{\mathbf{u}} f(1, 2) = (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -1 - \frac{\sqrt{3}}{2}$$

$$\approx -1.866.$$ See Figure 13.46.

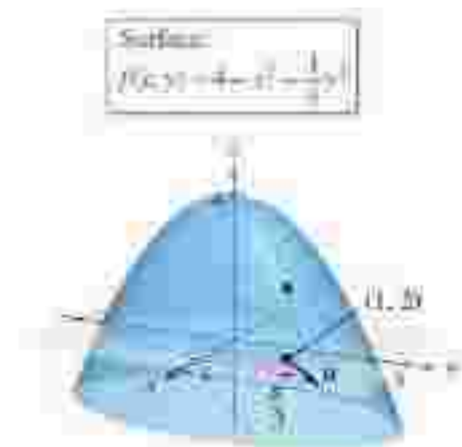


Figure 13.46

# The Gradient of a Function of Two Variables

# The Gradient of a Function of Two Variables

The **gradient** of a function of two variables is a vector-valued function of two variables.

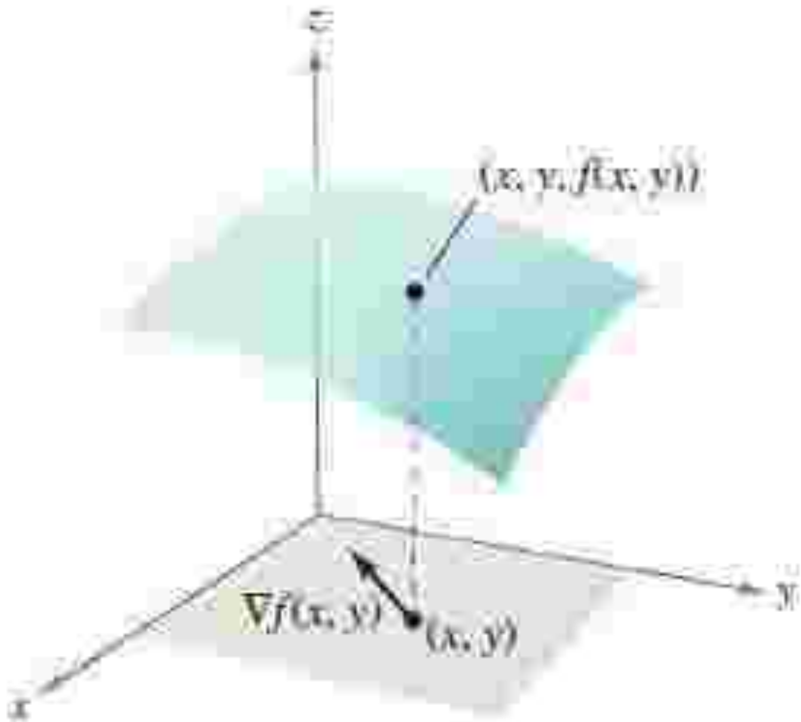
**Definition of Gradient of a Function of Two Variables**

Let $z = f(x, y)$ be a function of $x$ and $y$ such that $f_x$ and $f_y$ exist. Then the **gradient of** $f$, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

(The symbol $\nabla f$ is read as "del $f$.") Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each $(x, y)$, the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

# The Gradient of a Function of Two Variables




The gradient of $f$ is a vector in the $xy$-plane.

**Figure 13.48**

# Example 3 – *Finding the Gradient of a Function*

Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point (1, 2).

Solution:

Using

$$f_x(x, y) = \frac{y}{x} + y^2 \qquad \text{and} \qquad f_y(x, y) = \ln x + 2xy$$

you have

$$\nabla f(x, y) = \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.$$

At the point (1, 2), the gradient is

$$\nabla f(1, 2) = \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j}$$

$$= 6\mathbf{i} + 4\mathbf{j}.$$

# The Gradient of a Function of Two Variables

Because the gradient of *f* is a vector, you can write the directional derivative of *f* in the direction of **u** as

$$D_{\mathbf{u}}f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot [\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}].$$

In other words, the directional derivative is the dot product of the gradient and the direction vector.

**THEOREM 13.10 Alternative Form of the Directional Derivative**

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

# Example 4 – *Using ∇f(x, y) to Find a Directional Derivative*

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

Solution:

Because the partials of *f* are continuous, *f* is differentiable and you can apply Theorem 13.10.

A vector in the specified direction is

$$\overrightarrow{PQ} = \mathbf{v} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j}$$

# Example 4 – *Solution*

cont'd

$$= \frac{3}{4}\mathbf{i} + \mathbf{j}$$

and a unit vector in this direction is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}. \qquad \text{Unit vector in direction of } \overrightarrow{PQ}$$

Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}. \qquad \text{Gradient at } \left(-\frac{3}{4}, 0\right)$$

# Example 4 – *Solution*

cont'd

Consequently, at $\left(-\frac{3}{4}, 0\right)$ the directional derivative is

$$D_{\mathbf{u}}f\left(-\frac{3}{4},0\right) = \nabla f\left(-\frac{3}{4},0\right)\cdot\mathbf{u}$$

$$= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right)\cdot\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$$

$$= -\frac{27}{10}. \qquad \text{Directional derivative at } \left(-\tfrac{3}{4},0\right)$$

See Figure 13.49.

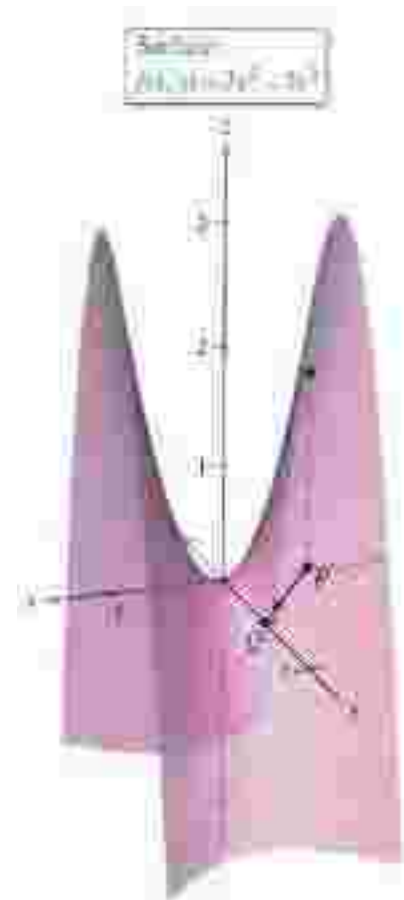


Figure 13.49

# Applications of the Gradient

# Applications of the Gradient

**THEOREM 13.11 Properties of the Gradient**

Let $f$ be differentiable at the point $(x, y)$.

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of $f$ is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is

   $\|\nabla f(x, y)\|.$ Maximum value of $D_{\mathbf{u}} f(x, y)$
3. The direction of *minimum* increase of $f$ is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is

   $-\|\nabla f(x, y)\|.$ Minimum value of $D_{\mathbf{u}} f(x, y)$

# Example 5 – *Finding the Direction of Maximum Increase*

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where $x$ and $y$ are measured in centimeters. In what direction from (2, –3) does the temperature increase most rapidly? What is this rate of increase?

Solution:

The gradient is

$$\begin{aligned}\nabla T(x, y) &= T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j} \\ &= -8x\mathbf{i} - 2y\mathbf{j}.\end{aligned}$$

# Example 5 – *Solution*

cont'd

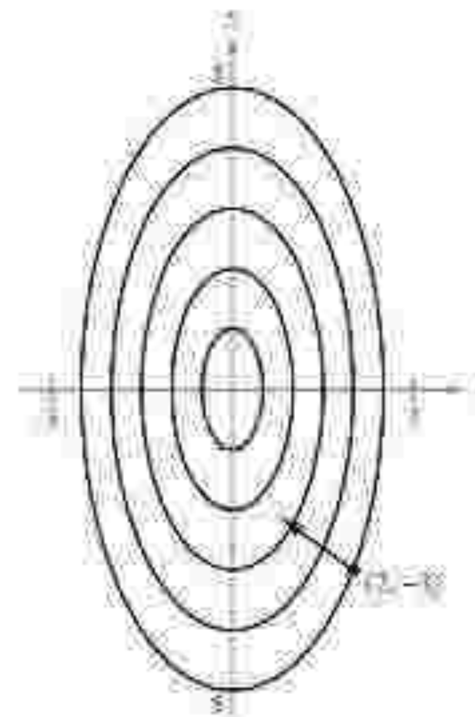
It follows that the direction of maximum increase is given by

$$\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$$

as shown in Figure 13.51, and the rate of increase is

$$\begin{aligned} \|\nabla T(2, -3)\| &= \sqrt{256 + 36} \\ &= \sqrt{292} \\ &\approx 17.09^\circ \text{ per centimeter.} \end{aligned}$$

Level curves:
$T(x, y) = 20 - 4x^2 - y^2$



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.

**Figure 13.51**

# Applications of the Gradient

**THEOREM 13.12 Gradient Is Normal to Level Curves**

If $f$ is differentiable at $(x_0, y_0)$ and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through $(x_0, y_0)$.

# Example 7 – *Finding a Normal Vector to a Level Curve*

Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.

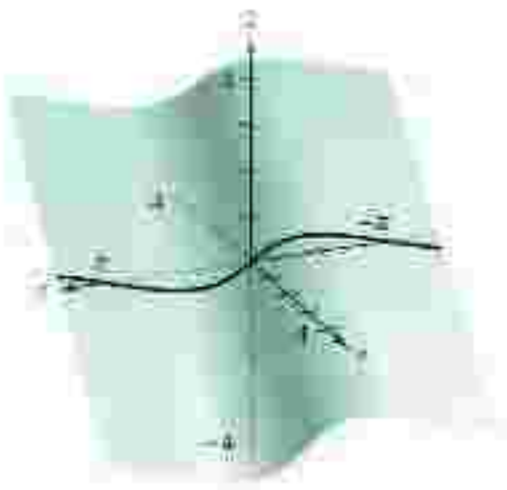
Solution:

The level curve for $c = 0$ is given by

$$0 = y - \sin x$$

$$y = \sin x$$

as shown in Figure 13.53(a).



The surface is given by $f(x, y) = y - \sin x$.

**Figure 13.53(a)**

# Example 7 – *Solution*

cont'd

Because the gradient vector of $f$ at $(x, y)$ is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$
$$= -\cos x\mathbf{i} + \mathbf{j}$$

you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point $(x, y)$.

Some gradient vectors are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j}$$

# Example 7 – *Solution*

cont'd

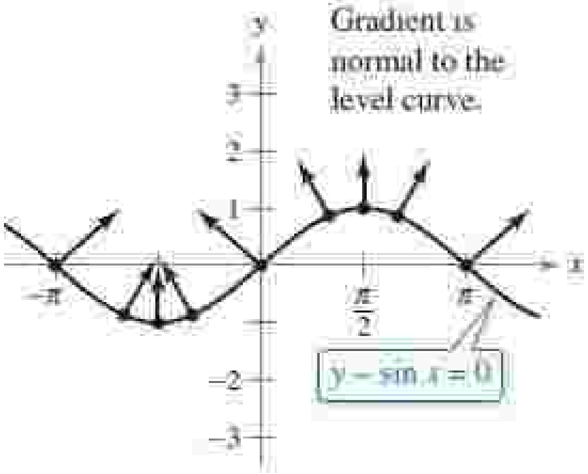
$$\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}.$$

These are shown in Figure 13.53(b).




(b) The level curve is given by $f(x, y) = 0$.

The level curve is given by $f(x, y) = 0$.

**Figure 13.53(b)**

## Functions of Three Variables

# Functions of Three Variables

**Directional Derivative and Gradient for Three Variables**

Let $f$ be a function of $x$, $y$, and $z$, with continuous first partial derivatives. The **directional derivative of $f$** in the direction of a unit vector

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of $f$** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of $f$ is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $\|\nabla f(x, y, z)\|$.   Maximum value of $D_{\mathbf{u}} f(x, y, z)$

4. The direction of *minimum* increase of $f$ is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $-\|\nabla f(x, y, z)\|$.   Minimum value of $D_{\mathbf{u}} f(x, y, z)$

## Example 8 – *Finding the Gradient for a Function of Three Variables*

Find $\nabla f(x, y, z)$ for the function given by

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of $f$ at the point $(2, -1, 1)$.

Solution:

The gradient vector is

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

$$= 2x\mathbf{i} + 2y\mathbf{j} - 4\mathbf{k}$$

# Example 8 – *Solution*

cont'd

So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$\nabla f(2, -1, 1) = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$.

See Figure 13.54.

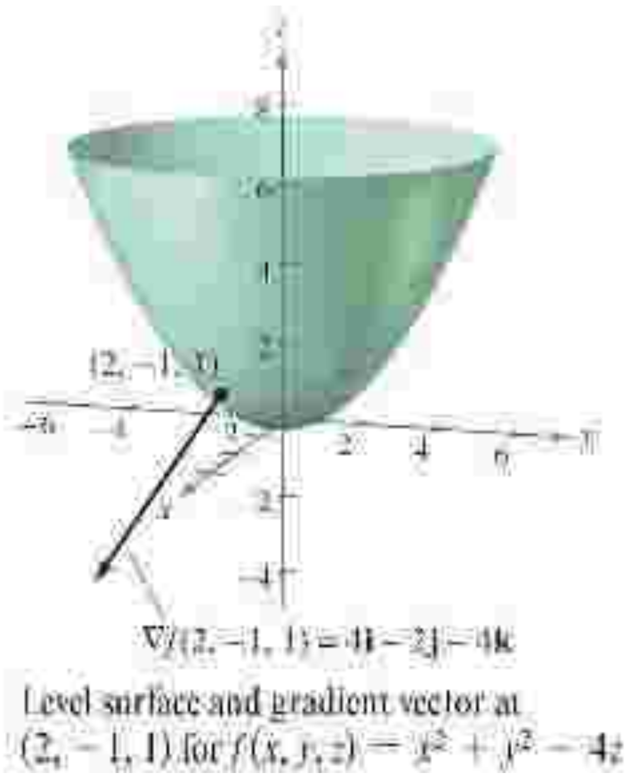



**Figure 13.54**

# Curl and Divergence

# Curl

If $\boldsymbol{F} = P\,\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\widehat{\boldsymbol{k}}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of *P*, *Q*, and *R* all exist, then the **curl** of **F** is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

# Curl

It has meaning when it operates on a scalar function to produce the gradient of *f*:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field **F** as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

# Curl

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \text{curl}\,\mathbf{F}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

**2**

$$\boxed{\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}}$$

# Example 1

If $\boldsymbol{F}(x, y, z) = xz\,\boldsymbol{i} + xyz\,\boldsymbol{j} - y^2\,\boldsymbol{k}$, find $curl\,\boldsymbol{F}$.

Solution:

Using Equation 2, we have

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

# Example 1 – *Solution*

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

# Curl

Recall that the gradient of a function *f* of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl.

The following theorem says that the curl of a gradient vector field is **0**.

$\boxed{3}$ **Theorem** If $f$ is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

# Curl

Since a conservative vector field is one for which $\boldsymbol{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\boldsymbol{F}$ is conservative, then $curl\ \boldsymbol{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

# Curl

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, “has no hole.”)

$\boxed{4}$ **Theorem** If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

# Curl

The reason for the name *curl* is that the curl vector is associated with rotations.

Another occurs when **F** represents the velocity field in fluid flow. Particles near $(x, y, z)$ in the fluid tend to rotate about the axis that points in the direction of $curl\ \boldsymbol{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1).

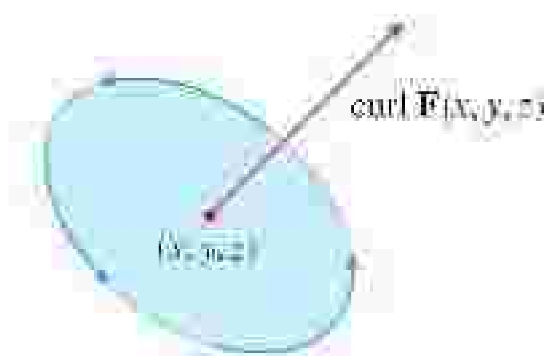



**Figure 1**

# Curl

If curl **F** = **0** at a point *P*, then the fluid is free from rotations at *P* and **F** is called **irrotational** at *P*.

In other words, there is no whirlpool or eddy at *P*.

If curl **F** = **0**, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If curl **F** ≠ **0**, the paddle wheel rotates about its axis.

# Divergence

# Divergence

If **F** = $P$ **i** + $Q$ **j** + $R$ **k** is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

$$\boxed{9} \qquad \boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Observe that curl **F** is a vector field but div **F** is a scalar field.

# Divergence

In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of $\nabla$ and **F**:

$$\boxed{10} \qquad \boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}}$$

# Example 4

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} + y^2\,\mathbf{k}$, find div $\mathbf{F}$.

Solution:

By the definition of divergence (Equation 9 or 10) we have

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

# Divergence

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence.

The next theorem shows that the result is 0.

**11 Theorem** If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $P$, $Q$, and $R$ have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

# Divergence

If **F**($x$, $y$, $z$) is the velocity of a fluid (or gas), then div **F**($x$, $y$, $z$) represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point ($x$, $y$, $z$) per unit volume.

In other words, div **F**($x$, $y$, $z$) measures the tendency of the fluid to diverge from the point ($x$, $y$, $z$).

If div **F** = 0, then **F** is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field $\nabla f$.

# Divergence

If *f* is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

# Divergence

We can also apply the Laplace operator $\nabla^2$ to a vector field

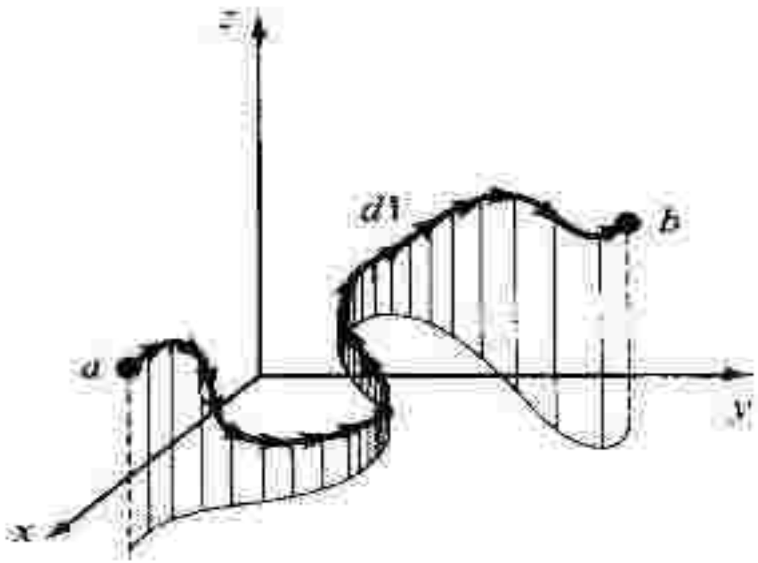
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

# Line, Surface and Volume Integrals

# Line Integral

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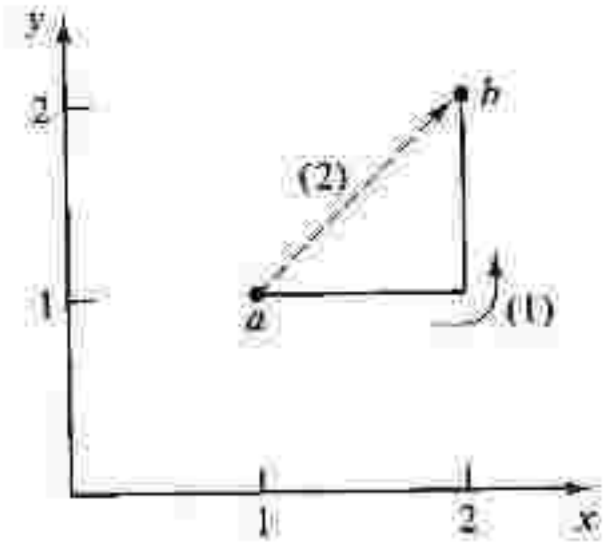
# Example



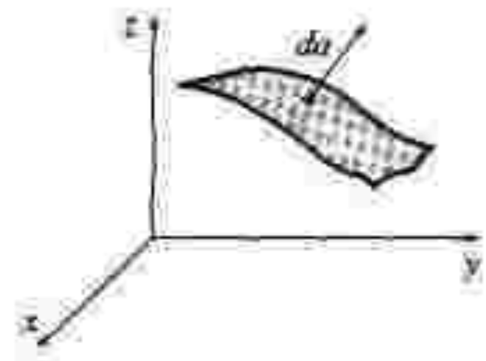
# Surface Integral

$\mathbf{v} \quad d\mathbf{a} \qquad \text{flux} \;,$

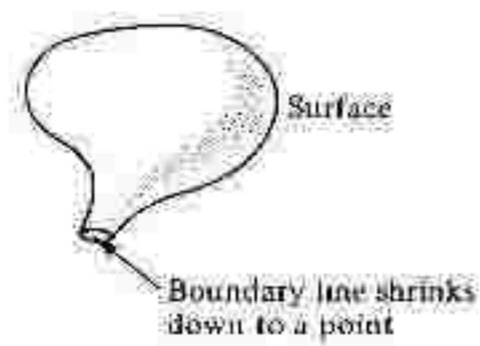
S

$|d\mathbf{a}|$ is an infinitesimal patch of the surface,

$d\mathbf{a}$ is perpendicular to this patch.




For a given boundary line there many different surfaces, on which the surface integral depends. It is independent only if

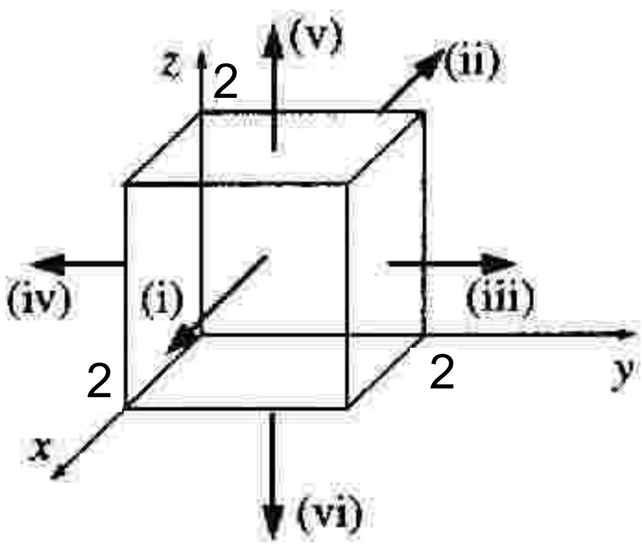



If the surface is closed: ○

# Example

$$(2xz\hat{\mathbf{x}} \quad (x \quad 2)\hat{\mathbf{y}} \quad y(z^2 \quad 3)\hat{\mathbf{z}})\; d\mathbf{a}$$

$exclude(vi)$



# Volume Integral

$$T(x, y, z)d \qquad d \qquad dxdydz$$

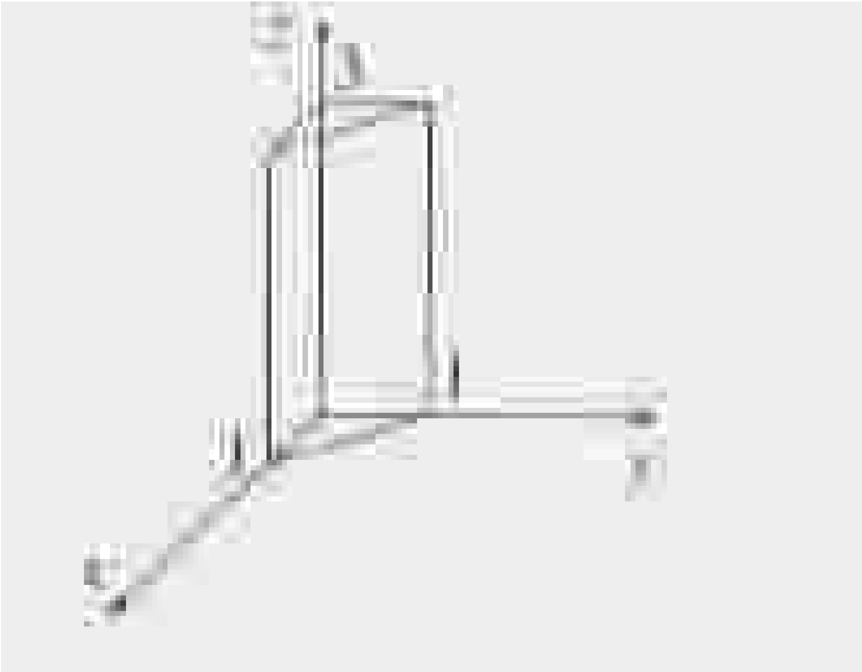
V

$$\mathbf{v}d \qquad \hat{\mathbf{x}} \quad v_x d \qquad \hat{\mathbf{y}} \quad v_y d \qquad \hat{\mathbf{z}} \quad v_z d$$

# Example

$$xyz^2 d$$

*prism*



# Fundamental Theorem for Gradients

Generalization of work done by conservative force

$$\int_{\mathbf{a}\,P}^{\mathbf{b}} \nabla T \cdot d\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a}), \quad \text{if } \mathbf{a} \to \mathbf{b} \quad \oint_P \nabla T \cdot d\mathbf{r} = 0$$

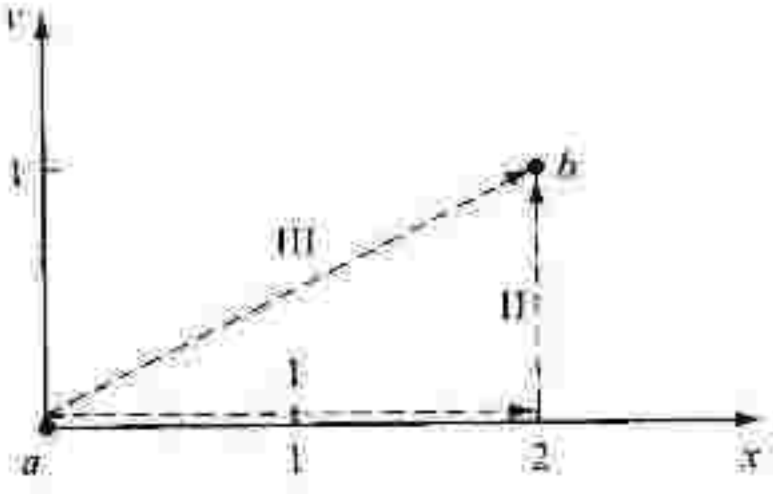
The line integral does not depend on the path P.

Example : Mechanical work

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V \cdot d\mathbf{r} = -\left(V(\mathbf{b}) - V(\mathbf{a})\right)$$

# Example 1.9

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla(xy^2)\ dr$$ along path $I, II$ and $III$

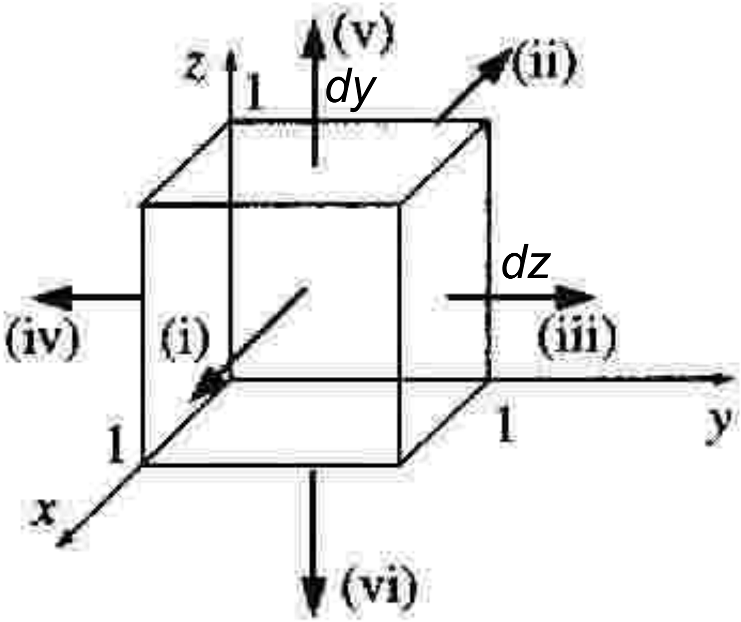


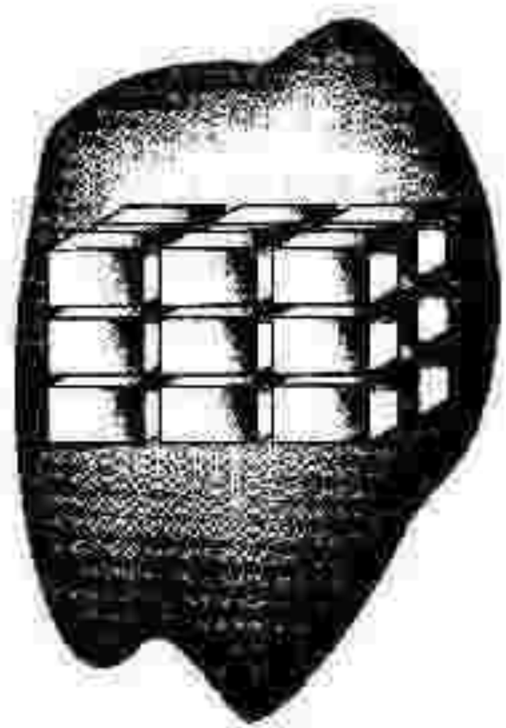
# Fundamental Theorem for Divergences

Gauss's Divergence Theorem

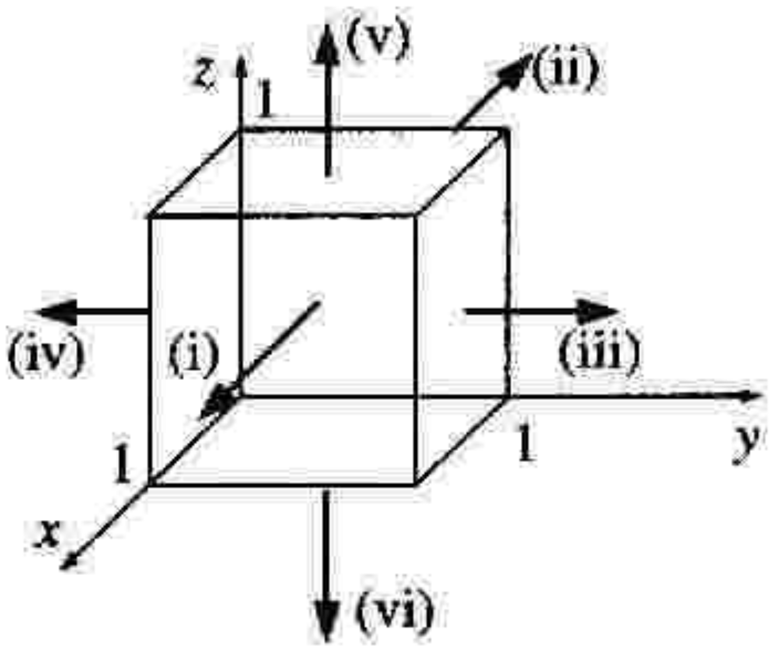
$$_{V}\ (\nabla \ \ \mathbf{v})d \qquad \circ\, \mathbf{v} \ \ d\mathbf{a}$$
$$_{S}$$

The surface S encloses the volume V.


(v)
(ii)
z
1
dy
dz
(iv)
(i)
(iii)
1
1
y
x
(vi)



# Example



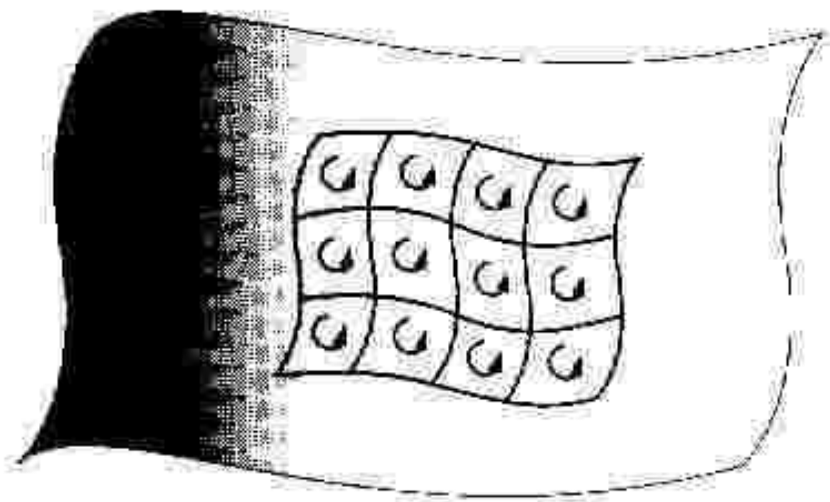

Check the divergence theorem for

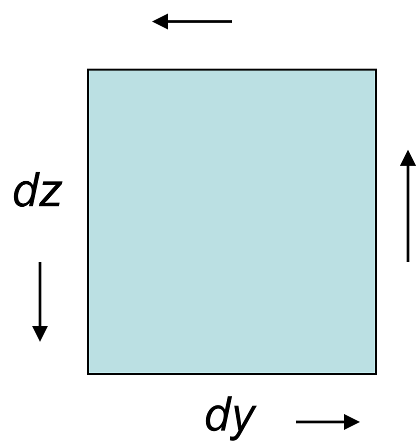
$$\mathbf{v} \quad y^2\hat{\mathbf{x}} \quad (2xy \quad z^2)\hat{\mathbf{y}} \quad (2xy)\hat{\mathbf{z}}$$

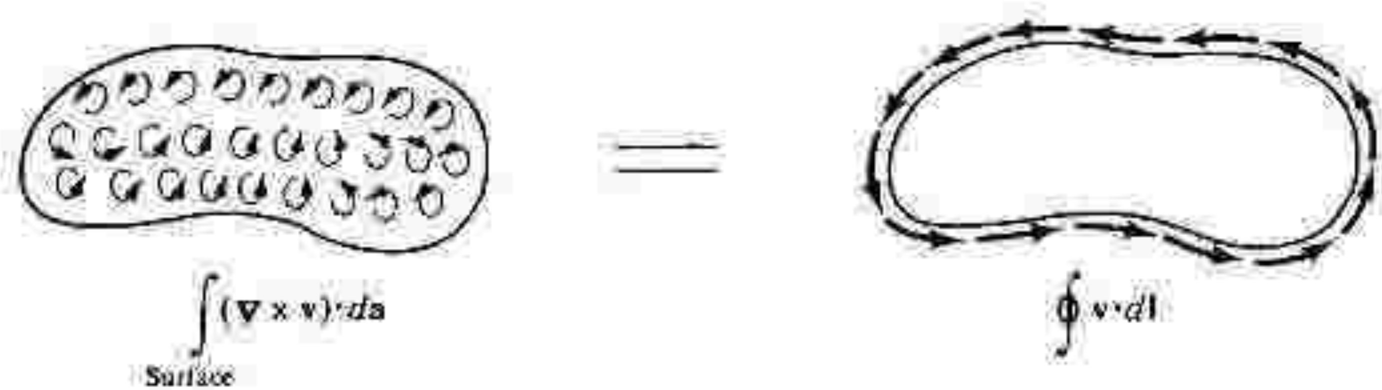
# Fundamental Theorem for Curls

Stokes' theorem

The path P is the boundary of the surface S.
The integral does not depend on S.




dz
dy





You must do it in a consistent way!

# Example

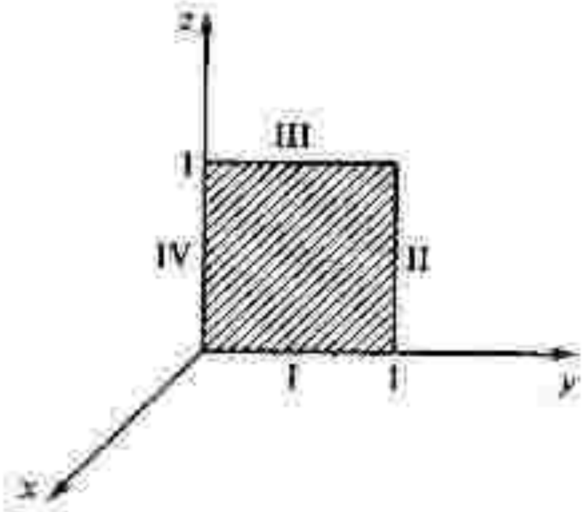



Figure 1.34

Check Stokes' Theorem for

# THANK YOU

# *Laplace Transform & Fourier Series*

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January 13, 2009

# Plan of Talk

# Objectives

After going through this unit the reader should be able to understand the

- Calculation of Laplace transform, which include the existence of the transform with some remarks on the theory.
- Properties of Laplace transform, transform of derivatives and integrals and convolution theorem which are crucial in the application of the method to the solution of ordinary differential equations.
- The use of unit step function as discontinuous forcing function which is very common in science and engineering.
- Dirac - Delta function ( concept of impulse, may be interrupted as a force of very large magnitude applied for just an instant).
- Laplace transform of periodic functions.
- Use of Laplace transform to solve the certain type of differential equations.
- Expansion of the periodic functions as a Fourier series.

# Plan of Talk

# Laplace Transform

**Definition**

The Laplace transform of a function $f(t)$ of a real variable $t$ and defined for $t \geq 0$ is a function $F(s)$ defined by $F(s) = \int_0^\infty e^{-st} f(t)dt$ provided the integral exists and symbolically this is written as $L[f(t)] = F(s)$.

# Laplace Transform of some basic functions

| $f(t)$ | $L[f(t)]$ | $f(t)$ | $L[f(t)]$ |
|---|---|---|---|
| $1$ | $\frac{1}{s}, \quad s > 0$ | $\sin at$ | $\frac{a}{s^2 + a^2}, \quad s > 0$ |
| $t$ | $\frac{1}{s^2}, \quad s > 0$ | $\cos at$ | $\frac{s}{s^2 + a^2}, \quad s > 0$ |
| $t^n$ | $\frac{n!}{s^{n+1}}, \quad s > 0, n = 0, 1, 2, \cdots$ | $\sinh at$ | $\frac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$ |
| $t^a$ | $\frac{\Gamma(a+1)}{s^a + 1}$ | $\cosh at$ | $\frac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$ |
| $e^{at}$ | $\frac{1}{s - a}, \quad s > a$ | | |

# Plan of Talk

# Existence of Laplace Transform

Here we discuss the condition of convergence of the improper integral

$$\int_0^\infty e^{-st} f(t)dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)dt \tag{1}$$

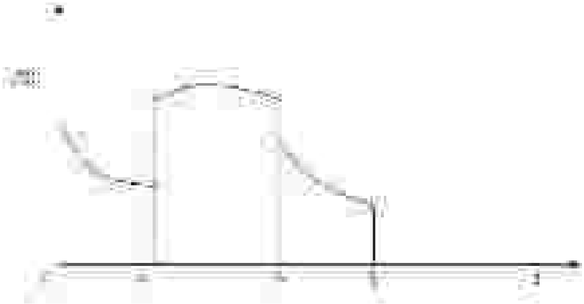
This integral converges whenever

$$\int_0^\infty |e^{-st} f(t)|dt \tag{2}$$

converges and in this case we say that the integral (1) converges absolutely and hence converges. If there exists a nonnegative function $g(t)$ such that $|e^{-st} f(t)| \leq g(t)$ and $\displaystyle\int_0^\infty g(t)dt$ converges, then by comparison test, it is concluded that (1) converges. Moreover, the function $f(t)$ defined for all $t \geq 0$ must be piecewise continuous on $[0, \infty)$, i.e. the function $f(t)$ is continuous over every finite interval $0 \leq t \leq b$, except at a finite number of points where there are jump discontinuity at which the left hand limit and the right hand limit exist but are unequal.

# Existence of Laplace Transform

The piecewise continuous function can be illustrated in the figure given below.



# Exponential order of $f(t)$

Apart from the piecewise continuity of $f(t)$ for $t \geq 0$, the other assumption is that $f(t)$ is of exponential order.

**Definition**

A function $f(t)$ on $[0, \infty)$ is said to be of exponential order if there exist constants $M$ and $k$ such that

$$|f(t)| \leq Me^{kt}, \ t \geq 0$$

The constant function, the function $t^n$, $e^t$, $\sin t$, $t^n \sin t$ where $n$ is a positive integer are of exponential order whereas $e^{t^2}$ is not of exponential order.

**Results:** Let $f$ be a piecewise continuous function on $[0, \infty)$ then

- $f$ is of exponential order if for some constant $\alpha$ $\lim_{t \to \infty} \left[\frac{f(t)}{e^{\alpha t}}\right] = 0$
- $f$ is not of exponential order if $\lim_{t \to \infty} \left[\frac{f(t)}{e^{\alpha t}}\right] = \infty$, for all real numbers $\alpha$.

# Example 1

Show that the function $f(t) = t^n$ is of exponential order.

If $(t)$ is of exponential order then we known that for every $\alpha > 0$

$$\lim_{t \to \infty} \left[ \frac{t^n}{e^{\alpha t}} \right] = 0$$

Hence for every given $\epsilon$ there exists some constant $t > t_0$ such that

$$\left| \frac{t^n}{e^{\alpha t}} - 0 \right| < \epsilon \text{ for all } t > t_0$$

Since $t^n$ is bounded on $[0, t_0]$, this implies that $|t^n| < Me^{\alpha t}$ for $t \geq 0, \alpha > 0$ on taking $\epsilon = 1$.
Hence by the definition $t^n$ is of exponential order.

# Existence of Laplace Transform

**Theorem**

*Let $f(t)$ be a piecewise continuous function on every finite interval $t \geq 0$ and of exponential order. Then there exist a real number $k$ such that $\int_0^\infty e^{-st} f(t) dt$ converges for $s > k$.*

# Example 2

Show that the Laplace transform of $f(t) = t^{-1/2}$ exists, even though it has the discontinuity of infinite order at $t = 0$.

Since $f(t)$ has the discontinuity of infinite order at $t = 0$, which is different from piecewise continuity. By definition of Laplace transform, we get

$$L[t^{-1/2}] = \int_0^\infty e^{-st} t^{-1/2} dt$$

On substituting $st = x$, we get

$$L[t^{-1/2}] = \int_0^\infty \frac{e^{-x}}{s} \left(\frac{x}{s}\right)^{-1/2} dx = \frac{\Gamma(1/2)}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

# Plan of Talk

# Properties of Laplace Transform

In this section we discuss the following important properties of Laplace transform which are useful to find an easiest way of computing the Laplace transform without using the definition.

- Linearity of Laplace Transform
- First Shifting Theorem
- Multiplication by $t^n$
- Division by $t$

# Linearity of Laplace Transform

If $L[f(t)] = F(s)$and $L[g(t)] = G(s)$, then for any constant $a$ and $b$

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)] = aF(s) + bG(s)$$

**Example 3.** Find the Laplace transform of $f(t) = (t-1)^2 + \sin t$ using the linearity property.

$$L[(t-1)^2 + \sin t] = L[t^2 - 2t + 1 + \sin t]$$

$$\Rightarrow L[f(t)] = L[t^2] - 2L[t] + L[1] + L[\sin t]$$

$$\Rightarrow L[f(t)] = \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s} + \frac{1}{s^2+1}, \quad s > 0$$

# First Shifting Theorem

If $L[f(t)] = F(s),\ s > k$, then $e^{at}f(t)$ has the transform $F(s-a),\ s-a > k$ that is

$$L[e^{at}f(t)] = F(s-a)$$

**Example 4.** Find the Laplace transform of $e^{at}(t-1)^2$.

$$L[(t-1)^2] = L[t^2 - 2t + 1] = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

Thus,

$$L[e^{at}(t-1)^2] = \frac{2}{(s-a)^3} - \frac{1}{(s-a)^2} + \frac{1}{s-a}$$

# Multiplication by $t^n$

If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n F^n(s)$$

where $F^n(s)$ is the $n^{th}$ differentiation of the transform with respect to $s$.

**Example 5.** Find the Laplace transform of $t^2 \cosh 2t$.

$$\text{Let } f(t) = \cosh 2t, \text{ then } L[f(t)] = \frac{s}{s^2 - 4}$$

Thus,

$$L[t^2 \cosh 2t] = \frac{(-1)^2 d^2}{ds^2}\left(\frac{s}{s^2-4}\right) = \frac{2s^3 + 24s}{(s^2-4)^3}, \quad s > 2$$

# Division by $t$

If $L[f(t)] = F(s)$ and $\lim_{t \to 0} \dfrac{f(t)}{t}$ exists, then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tilde{s})d\tilde{s}$$

provided the integral on the right hand side exists.

**Example 6.** Find the Laplace transform of $\dfrac{\sin t}{t}$.

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty F(\tilde{s})d\tilde{s}, \text{ where } F(\tilde{s}) = L[\sin t] = \frac{1}{s^2+1}$$

Thus,

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{\tilde{s}^2+1}d(\tilde{s}) = \frac{\pi}{2} - \tan^{-1} s$$

# Plan of Talk

# Laplace Transform of Derivative

**Theorem**

*If $f(t)$ is continuous for $t \geq 0$ and of exponential order and also $f'(t)$ is piecewise continuous and of exponential order for $t \geq 0$ then*

$$L[f'(t)] = sL[f(t) - f(0)]$$

*Similarly, if $f(t), f'(t), f''(t), \cdots, f^{(n-1)}(t)$ are continuous for all $t \geq 0$ and of exponential order and $f^n(t)$ is piecewise continuous and of exponential order. Then*

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

# Example 7

Using the Laplace transform of the derivatives find the Laplace transform of $f(t) = \sin^2 t$.

We have $f(t) = \sin^2 t \;\Rightarrow f'(t) = 2\sin t\cos t = \sin 2t$. Also $f(0) = 0$.

$$L[f'(t)] = sL[f(t)] - f(0) \;\Rightarrow\; L[\sin 2t] = sL[f(t)] - 0$$

$$\frac{2}{s^2+4} = sL[f(t)] \;\Rightarrow\; L[f(t)] = \frac{2}{s(s^2+4)}$$

# Laplace Transform of Integrals

## Theorem

*If $L[f(t)] = F(s)$ and $f(t)$ is piecewise continuous function of exponential order, then*

$$L\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}.$$

*In general,*

$$L\left[\underbrace{\int_0^t \int_0^t \cdots \int_0^t f(u)dudu\cdots du}_{n-times}\right] = \frac{F(s)}{s^n}$$

# Example 8

Find the Laplace transform of $\int_0^t e^{-t} \cos t dt$.

We know that $L[e^{-t} \cos t] = \dfrac{s+1}{(s+1)^2+1}$ (Using the first shifting theorem)

Thus, $L\left[\int_0^t f(u)du\right] = \dfrac{F(s)}{s}$

$$\text{Hence } L\left[\int_0^t e^{-t} \cos t dt\right] = \frac{s+1}{s[(s+1)^2+1]}$$

# Plan of Talk

# Unit Step Function

## Definition

The unit step function $u(t)$ is defined by

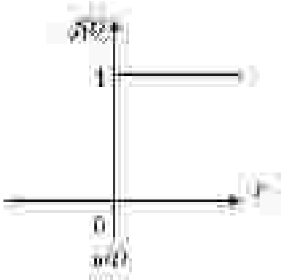
$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 1 \end{cases}$$

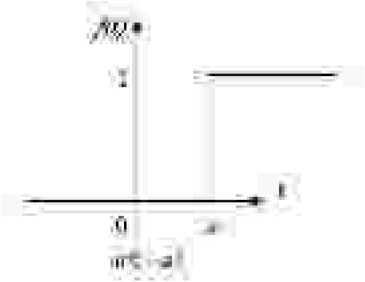
where 0 is the point of jump discontinuity. This function is also known as Heaviside function. If the jump discontinuity is at a point $t = a > 0$, then the unit step function $u(t-a)$ or $u_a(t)$ is defined by

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

# Graph of Unit Step Function

The Unit step function can be plotted as



# Second Shifting Theorem or T-Shifting

**Theorem**

*Let $f(t)$ be piecewise continuous function and of exponential order and if $L[f(t)] = F(s)$, then*

$$L[u(t-a)f(t-a)] = e^{-as}F(s)$$

**Example 9.** Find the Laplace transform of $e^{t-3}u(t-3)$.

On comparing $e^{t-3}u(t-3)$ with $f(t-a)u(t-a)$, we get that

$L[e^{t-3}u(t-3)] = e^{-3s}F(s)$, where $F(s) = L[f(t)]$ and $f(t) = e^{t}$ so

$$F(s) = \frac{1}{s-1}$$

Thus,

$$L[e^{t-3}u(t-3)] = \frac{e^{-3s}}{s-1}$$

# Example 10

Express the following function in terms of unit step function and then find the Laplace transform

$$f(t) = \begin{cases} 2t & \text{if } 0 < t < \pi \\ 1 & \text{if } t > \pi \end{cases}$$

We can write the function $f(t)$ as

$$f(t) = 2t[u(t-0) - u(t-\pi)] + 1[u(t-\pi)]$$

Thus, we get $f(t) = 2tu(t) - 2(t-\pi)u(t-\pi) - (2\pi - 1)u(t-\pi)$

On taking the Laplace transform on both sides and using the second shifting property we get

$$L[f(t)] = \frac{2}{s^2} - 2\frac{e^{-\pi}}{s^2} - (2\pi - 1)\frac{e^{-\pi}}{s}$$

# Unit Impulse Function or Dirac Delta Function

## Definition

The unit impulse function or Dirac Delta function is given by $\delta(t-a) = \lim_{k \to 0} f_k(t-a)$, where

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

and $\int_0^\infty \delta(t-a)dt = 1$

**Remark.** The function $\delta(t-a)$ is zero everywhere except at a single point but the integral of this function from 0 to $\infty$ is one, while in calculus integral of such functions are zero.

# Laplace Transform of Dirac Delta Function

Since

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}$$

Thus it can be written in terms of unit step function as

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-(a+k))]$$

Taking Laplace transform on both sides

$$L[f_k(t-a)] = \frac{1}{k}[L[u(t-a)] - L[u(t-(a+k))]] =$$
$$\frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s} = e^{-as}\frac{1-e^{-ks}}{ks}\right]$$

To find the Laplace transform of Dirac Delta function take the limit $k \to 0$. Thus, $L[\delta(t-a)] = e^{-as}$. In particular $L[\delta(t)] = 1$.

# Example 11

Find the Laplace transform of $t^2 u(t-1) + \delta(t-1)$.

$L[t^2 u(t-1) + \delta(t-1)] = L[(t-1+1)^2 u(t-1) + L\delta(t-1)]$

$= L[(t-1)^2 u(t-1)] + 2L[(t-1)u(t-1)] + L[u(t-1)] + L[\delta(t-1)]$

$$= e^{-s}\frac{2}{s^2} + 2\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + e^{-s}$$

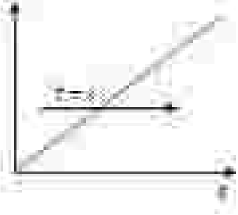
$$= \frac{e^{-s}}{s^3}[2 + 2s + s^2 + s^3]$$

# Plan of Talk

# Convolution of Functions

Let $f(t)$ and $g(t)$ be the functions defined for $t \geq 0$. Then the convolution of $f(t)$ and $g(t)$ is defined by

$$(f * g) = \int_0^t f(\tau)g(t-\tau)d\tau, \ t \geq 0$$



For example the convolution of $e^{3t}$ and $\sin 4t$ is

$$e^{3t} * \sin 4t = \int_0^t e^{3t} \sin 4(t-\tau)d\tau.$$

# Convolution Theorem

**Theorem**

*If* $L[f(t)] = F(s)$ *and* $L[g(t)] = G(s)$*, then*

$$L[f * g] = L[f(t)]L[g(t)]$$

**Example 12.** Find the Laplace transform of $t * e^{at}$.

$$L[t * e^{at}] = L[t]L[e^{at}] = \frac{1}{s^2} \times \frac{1}{s-a} = \frac{1}{s^2(s-a)}$$

# Error Function

## Definition

Error function is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

Naturally, the complementary error function is
$erfc(t) = 1 - erf(t) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_t^\infty e^{-x^2} dx.$

# Plan of Talk

# Laplace Transform of Periodic Function

**Definition**

A function $f(t)$ is said to be periodic if $f(t+a) = f(t)$ for all values of $a$. The number $a$ is called the period of $f(t)$.

i) Sawtoothwave function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < a \\ f(t+a) & \text{otherwise} \end{cases}$$

ii) Squarewave function

$$f(t) = \begin{cases} k & \text{if } 0 \le t < a \\ -k & \text{if } a \le t < 2a \end{cases}$$

# Laplace Transform of Periodic Function

## Theorem

*If $f(t)$ is piecewise continuous function and of exponential order and periodic with period $a$. Then*

$$L[f(t)] = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)dt$$

# Example 13

Find the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{if } a < t < \frac{a}{2} \\ -1 & \text{if } \frac{a}{2} < t < a \end{cases}$$

$$L[f(t)] = \frac{1}{1 - e^{-as}} \left[ \int_0^a e^{-st} f(t) dt \right]$$

$$= \frac{1}{1 - e^{-as}} \left[ \int_0^{\frac{a}{2}} e^{-st} f(t) dt - \int_{\frac{a}{2}}^a e^{-st} f(t) dt \right]$$

$$= \frac{1}{s(1 - e^{-as})} \left[ 1 + e^{-as} - 2e^{\frac{-as}{2}} \right]$$

# Plan of Talk

# Inverse Laplace Transform

## Definition

If $L[f(t)] = F(s)$, then the function $f(t)$ is called the inverse Laplace transform of $F(s)$. In symbolic way, we write

$$\text{If } L[f(t)] = F(s) \text{ then } f(t) = L^{-1}[F(s)]$$

Some of the standard results are given below

- $L^{-1}\left[\dfrac{1}{s}\right] = 1$
- $L^{-1}\left[\dfrac{n!}{s^{n+1}}\right] = t^n$
- $L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}$
- $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$
- $L^{-1}\left[\dfrac{a}{s^2+a^2}\right] = \sin at$

# Some More Important Results

Some more results using the properties of Laplace transform are given below

If $L^{-1}[F(s)] = f(t)$, then

- $L^{-1}[F(s-a)] = e^{at}f(t)$
- $L^{-1}[F^n(s)] = (-1)^n t^n f(t)$
- $L^{-1}\left[\int_s^\infty F(\tilde{s})d\tilde{s}\right] = \frac{f(t)}{t}$
- $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)du$
- $L^{-1}[e^{-as}F(s)] = u(t-a)f(t-a)$
- $L^{-1}[F(s)G(s)] = f(t) * g(t)$

# Example 14

Find the inverse Laplace transform of $\dfrac{s+2}{s^2-4s+13}$.

Since $\dfrac{s+2}{s^2-4s+13} = \dfrac{s+2}{(s-2)^2+3^2} = \dfrac{s-2+4}{(s-2)^2+3^2}$

Thus,

$$L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right] = L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + L^{-1}\left[\frac{4}{(s-2)^2+3^2}\right]$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

# Example 15

Find the inverse Laplace transform of $\dfrac{s}{(s^2-1)^2}$.

Since $\dfrac{s}{(s^2-1)^2} = -\dfrac{1}{4(s+1)^2} + \dfrac{1}{4(s-1)^2}$

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = -\frac{1}{4}L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= -\frac{1}{4}e^{t}.t + \frac{1}{4}t.e^{-t}$$

$$= \frac{t}{4}(e^{t} - e^{-t}) = \frac{t}{2}\sinh t$$

# Example 16

Find the inverse Laplace transform of $\log \dfrac{1+s}{s}$.

Let $L^{-1}\left[\log \dfrac{1+s}{s}\right] = f(t)$. Thus, $F(s) = \log \dfrac{1+s}{s}$

We know that

$$L[tf(t)] = -\frac{d}{ds}F(s) = -\frac{d}{ds}[\log(1+s) - \log s]$$

$$= -\frac{1}{1+s} + \frac{1}{s} = -L[e^{-t}] + L[1] = L[1 - e^{-t}]$$

$$\Rightarrow tf(t) = 1 - e^{-t} \;\Rightarrow\; f(t) = \frac{1 - e^{-t}}{t}$$

# Example 17

Using the convolution theorem find the inverse Laplace transform of $\dfrac{s^2}{(s^2+4)^2} = \dfrac{s}{(s^2+4)} \cdot \dfrac{s}{(s^2+4)}$.

By convolution theorem we have $L^{-1}[F(s)G(s)] = f * g$. Here, $F(s) = G(s) = \dfrac{s}{s^2+4}$. Thus,

$$L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{s}{s^2+4}\right] = L^{-1}\left[\frac{s}{s^2+4}\right] * L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= \cos 2t * \cos 2t = \int_0^t \cos 2u \cos 2(t-u) du$$

$$= \frac{1}{2}\int_0^t [\cos 2u + \cos(4u - 2t)] du = \frac{1}{4}[2t\cos 2t + \sin 2t]$$

# Example 18

Find the inverse Laplace transform of $\dfrac{2}{s^2} - \dfrac{2e^{-2s}}{s^2} - \dfrac{4e^{-2s}}{s}$.

$$L^{-1}\left[\frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s}\right] = L^{-1}\left[\frac{2}{s^2}\right] - L^{-1}\left[\frac{2e^{-2s}}{s^2}\right] - L^{-1}\left[\frac{4e^{-2s}}{s}\right]$$

Using the second shifting property in inverse form i.e. $L^{-1}[e^{-as}F(s)] = u(t-a)f(t-a)$ for second and third term in the above equation we get

$$= 2t - 2(t-2)u(t-2) - 4u(t-2) = 2t - 2tu(t-2)$$

Hence

$$f(t) = \begin{cases} 2t & 0 < t < 2 \\ 0 & t > 2 \end{cases}$$

# Plan of Talk

# Solution of Differential Equations

Consider the initial value problem
$y'' + py' + qy = r(t),\ y(0) = y_0,\ y'(0) = y_0^{'}$ where $p,\ q$ are constants.
To find the particular solution of this differential equation, apply Laplace transform on both the sides of this equation to obtain

$$L[y''] + pL[y'] + qL[y] = L[r(t)]$$

$$\Rightarrow s^2L[y] - sy(0) - y'(0) + p[sL[y] - y(0)] + qL[y] = R(s),\ R(s) = L[r(t)]$$

Simplifying, we obtain

$$L[y] = \frac{R(s) + (s+p)y_0 + y_0^{'}}{s^2 + ps + q}$$

Since the RHS is a function of $s$, thus, we obtain

$$y(t) = L^{-1}\left[\frac{R(s) + (s+p)y_0 + y_0^{'}}{s^2 + ps + q}\right]$$

Here it has been assumed that the Laplace transforms of $r(t), y, y', y''$ exist.

# Example 19

solve the differential equation $y'' + 4y' + 3y = e^{-t},\ y(0) = 1,\ y'(0) = 1.$ On taking Laplace transform on both the sides, we obtain

$$s^2L[y] - sy(0) - y'(0) + 4[sL[y] - y(0)] + 3L[y] = L[e^{-t}]$$

$$(s^2 + 4s + 3)L[y] = s + 1 + 4s + \frac{1}{s+1}$$

$$\Rightarrow L[y] = \frac{s^2 + 6s + 6}{(s+1)^2(s+3)}$$

$$y = L^{-1}\left[\frac{s^2 + 6s + 6}{(s+1)^2(s+3)}\right] = L^{-1}\left[\frac{7}{4s+1} + \frac{1}{2(s+1)^2} - \frac{3}{4(s+3)}\right]$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$$

# Integral Equations

## Definition

The equation $f(t) = y(t) + \int_0^t y(u)g(t-u)du$ is called the integral equation in which the function $y(t)$ is unknown.

This equation is of special form because the integrand is the convolution of two functions and we solve this equations by means of Laplace transform.

$$L[f(t)] = L[y(t)] + L[y(t)]L[g(t)]$$

$$l[y] = \frac{L[f(t)]}{1 + L[g(t)]}$$

# Example 20

Solve the integral equation $y(t) = t^3 + \int_0^t y(u)\,\mathsf{sin}(t-u)du.$

Applying Laplace transform on both the sides,

$$L[y] = L[t^3] + L[y(t)]L[\mathsf{sin}\,t]$$

$$L[y] = \frac{L[t^3]}{1 - L[\mathsf{sin}\,t]} = \frac{3!}{s^4}\left(\frac{s^2+1}{s^2}\right)$$

$$= \frac{3!}{s^4} + \frac{3!}{s^6}$$

# Plan of Talk

# Introduction

**Definition**

A function $f(x)$ is said to be of period $p$ if

$$f(x+p) = f(x)$$

**Definition**

A function $f(x)$ defined on any symmetrical placed interval about origin is said to be even if

$$f(-x) = f(x)$$

and is said to be odd if

$$f(-x) = -f(x)$$

# Properties of Even & Odd Functions

The product of even and odd functions have the properties

- (even)(odd)=(odd)(even)=odd and (even)(even)=(odd)(odd)=even
- if $f(x)$ is odd, $\int_{-a}^{a} f(x)dx = 0$
- if $f(x)$ is even, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

# Fourier Series

## Definition

The series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{3}$$

with period $2\pi$ is called Fourier series if

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \ \ n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \ \ n = 1, 2, \cdots$$

# Example 21

Find the Fourier series of the function
$f(x) = x, \ -\pi \leq x \leq \pi, f(x + \pi) = f(x)$.

Here, $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)dx = 0$

$a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} x \cos nx dx = 0$

$b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} x \sin nx dx = \dfrac{2}{\pi} \displaystyle\int_{0}^{\pi} x \sin nx dx$

$= -\dfrac{2}{n} \cos n\pi = \dfrac{2}{n}(-1)^{n+1}$. Thus, we get

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$ which is the required series.

# Example 22

Find the Fourier series of

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

Here, $a_0 = \dfrac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_{0}^{\pi}(1)dx\right] = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\cos nx)dx + \frac{1}{\pi}\int_{0}^{\pi}(\cos nx)dx = 0$$

Similarly, $b_n = \dfrac{1}{\pi}\int_{-\pi}^{0}(-\sin nx)dx + \dfrac{1}{\pi}\int_{0}^{\pi}(\sin nx)dx$

$$= \frac{1}{n\pi}[\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0]$$

$$= \frac{2}{n\pi}(1 - \cos n\pi) = \frac{2}{n\pi}(1 - (-1)^n)$$

Thus, $b_{2n} = 0,\ b_{2n-1} = \dfrac{4}{(2n-1)\pi},\ n = 1, 2, \cdots$ and the corresponding Fourier series is given by

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

# Convergence and Sum of Fourier Series

If $f(x)$ is piecewise continuous function in the interval $-\pi \leq x \leq \pi$ with period $2\pi$ and also piecewise smooth (function $f(x)$ is said to be piecewise smooth if it is differential on the interval, except a finite number of points where left and right hand derivative exist but are not equal) then the Fourier series of $f(x)$ is convergent and the sum of the series is $f(x)$, except at point $x_0$ where $f(x)$ has jump discontinuity and the sum of the series is the average of left hand and right hand limit of $f(x)$ at $x_0$, that is $\dfrac{f(x_0^+) + f(x_0^-)}{2}$

# Fourier Series of any period $p = 2l$

The function $f(x)$ with period $p = 2l$ has the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

$$\text{where } a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \;\; n = 1, 2, \cdots$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \;\; n = 1, 2, \cdots$$

# Example 23

Find the Fourier series expansion of

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

Here $l = 2$. With $p = 2l$, we have

$$a_0 = \frac{1}{2}\left[\int_{-2}^{0} 0 dx + \int_{0}^{2} x dx\right] = 1$$

$$a_n = \frac{1}{2}\int_{0}^{2} \cos\frac{n\pi x}{2} dx = 0, \ n = 1, 2, \cdots$$

$$b_n = \frac{1}{2}\int_{0}^{2} \sin\frac{n\pi x}{2} dx = \frac{1-(-1)^n}{n\pi}, \ n = 1, 2, \cdots$$

Thus, the Fourier series is given by

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\pi\frac{x}{2}$$

# Half Range Fourier Expansions

## Theorem

*If $f(x)$ is integrable on the interval $[-l, l]$, then*

- *If $f(x)$ is even then its Fourier series contains only cosine terms and the coefficients are given by*

$$a_0 = \frac{1}{l}\int_0^l f(x)dx, \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, \quad n = 1, 2, \cdots$$

$$b_n = 0$$

- *If $f(x)$ is odd then its Fourier series contains only sine terms and the coefficients are given by*

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad n = 1, 2, \cdots$$

# Example 24

Find the Fourier *sine* series of the function $f(x) = x,\ 0 \leq x \leq \pi$.

For the Fourier *sine* series, we extend the function $f(x) = x$ for $0 \leq x \leq \pi$ as an odd function, i.e. $f(x) = x,\ -\pi \leq x \leq \pi$ which is a $2\pi$ periodic.

Then its Fourier series is given by

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right),\ 0 \leq x \leq \pi$$

# Example 25

Find the Fourier *cosine* series of the function $f(x) = x,\ 0 \le x \le \pi$.

For the Fourier *cosine* series, we extend the function $f(x)$ as an even function with period $2\pi$

$$f(x) = \begin{cases} x & -\pi \le x \le 0 \\ -x & 0 \le x \le \pi \end{cases}$$

Since it is an even function $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi x dx = \dfrac{\pi}{2}$,

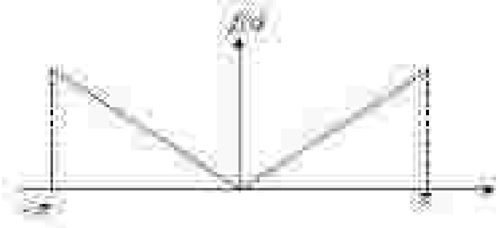
$a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi x \cos nx dx,\ n = 1, 2, \cdots$

Thus, we get $a_n = \dfrac{2}{\pi n^2}(\cos n\pi - 1) = \dfrac{2}{\pi n^2}((-1)^n - 1)$

# Example 25

Thus, the Fourier series is given by

$$x = \frac{\pi}{4} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \cdots\right), \ 0 \le x \le \pi$$



# Plan of Talk

# Summary

In this lecture we have discussed

- Calculation Laplace transform using the definition.
- Sufficient condition for the existence of the Laplace transform of the function is the function should be piecewise continuous and of exponential order.
- Properties of the Laplace transform of the functions.
- Unit step function, Dirac's delta function and convolution theorem.
- If a function is periodic with period $a$ then the Laplace transform of this functions is $\dfrac{1}{1-e^{-as}}\displaystyle\int_0^a e^{-st}f(t)dt.$
- Laplace transform can be used to solve the initial value problems using the Laplace transform of derivatives.
- Expansion of the periodic function in terms of Fourier series.

# THANKS

# UNIT – IV
# Fourier Transforms

# Fourier series

- To go from f($\theta$) to f($t$) substitute $\theta = \frac{2\pi}{T} t = \omega_0 t$

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

- To deal with the first basis vector being of length 2π instead of π, rewrite as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

# Fourier series

- The coefficients become

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(k\omega_0 t)\,dt$$

# Fourier series

- Alternate forms

$$f(t) \quad \frac{a_0}{2} \quad \overset{\infty}{\underset{n \quad 1}{}} a_n(\cos(n\omega_0 t) \quad \frac{b_n}{a_n}\sin(n\omega_0 t))$$

$$\frac{a_0}{2} \quad \overset{\infty}{\underset{n \quad 1}{}} a_n(\cos(n\omega_0 t) \quad \tan(\quad_n)\sin(n\omega_0 t))$$

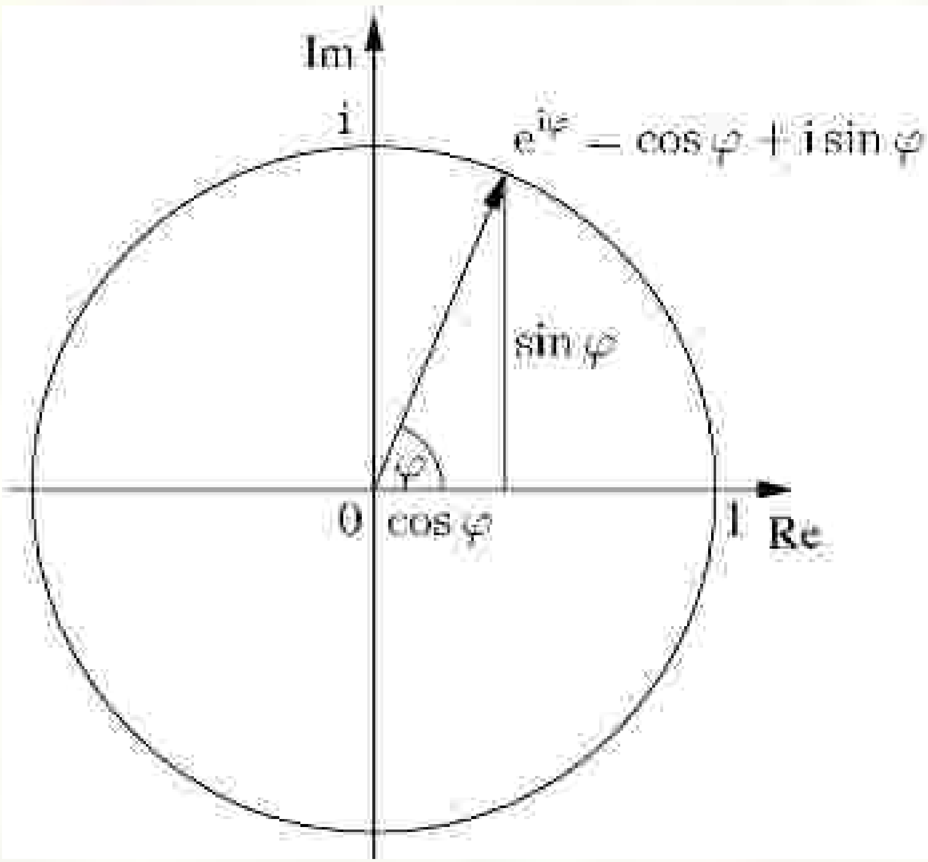
$$\frac{a_0}{2} \quad \overset{\infty}{\underset{n \quad 1}{}} c_n(\cos(n\omega_0 t) \quad\quad _n)$$

- where

$$c_n \quad \sqrt{a_n^2 \quad b_n^2} \quad \text{and} \quad\quad _n \quad\quad \tan^{\;1} \frac{b_n}{a_n}$$

# Complex exponential notation

- Euler's formula $e^{ix} \quad \cos(x) \quad i\sin(x)$




Phasor notation:

$$x \quad iy \quad |z|e^{i}$$

$$\text{where } |z| \quad \sqrt{x^2 \quad y^2}$$

$$\sqrt{z\bar{z}}$$

$$\sqrt{(x \quad iy)(x \quad iy)}$$

$$\text{and} \qquad \tan^{\ 1}\frac{y}{x}$$

# Euler's formula

- Taylor series expansions

$$e^{x} \quad 1 \quad x \quad \frac{x^{2}}{2!} \quad \frac{x^{3}}{3!} \quad \frac{x^{4}}{4!} \quad \ldots$$

- Even function ( f($x$) = f(-$x$) )

$$\cos(x) \quad 1 \quad \frac{x^{2}}{2!} \quad \frac{x^{4}}{4!} \quad \frac{x^{6}}{6!} \quad \frac{x^{8}}{8!} \quad \ldots$$

- Odd function ( f($x$) = -f(-$x$) )

$$\sin(x) \quad x \quad \frac{x^{3}}{3!} \quad \frac{x^{5}}{5!} \quad \frac{x^{7}}{7!} \quad \frac{x^{9}}{9!} \quad \ldots$$

$$e^{ix} \quad 1 \quad ix \quad \frac{x^{2}}{2!} \quad \frac{ix^{3}}{3!} \quad \frac{x^{4}}{4!} \quad \frac{ix^{5}}{5!} \quad \frac{x^{6}}{6!} \quad \frac{ix^{7}}{7!} \quad \ldots$$

$$\cos(x) \quad i\sin(x)$$

# Complex exponential form

- Consider the expression

$$f(t) \qquad \mathop{}_{n \quad \infty}^{\infty} F_n e^{in\omega_0 t} \qquad \mathop{}_{n \quad \infty}^{\infty} F_n \cos(n\omega_0 t) \quad iF_n \sin(n\omega_0 t)$$

$$\mathop{}_{n \;\; 0}^{\infty} (F_n \quad F_{\;\;n}) \cos(n\omega_0 t) \quad i(F_n \quad F_{\;\;n}) \sin(n\omega_0 t)$$

- So $a_n \quad F_n \quad F_{\;\;n}$ and $b_n \quad i(F_n \quad F_{\;\;n})$
- Since $a_n$ and $b_n$ are real, we can let $F_{\;\;n} \quad \overline{F_n}$

and get $a_n \quad 2\,\mathrm{Re}(F_n)$ and $b_n \quad 2\,\mathrm{Im}(F_n)$

$$\mathrm{Re}(F_n) \quad \frac{a_n}{2} \quad \text{and} \quad \mathrm{Im}(F_n) \quad \frac{b_n}{2}$$

# Complex exponential form

- Thus

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,dt - i\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)(\cos(n\omega_0 t)\,dt - i\sin(n\omega_0 t))dt$$

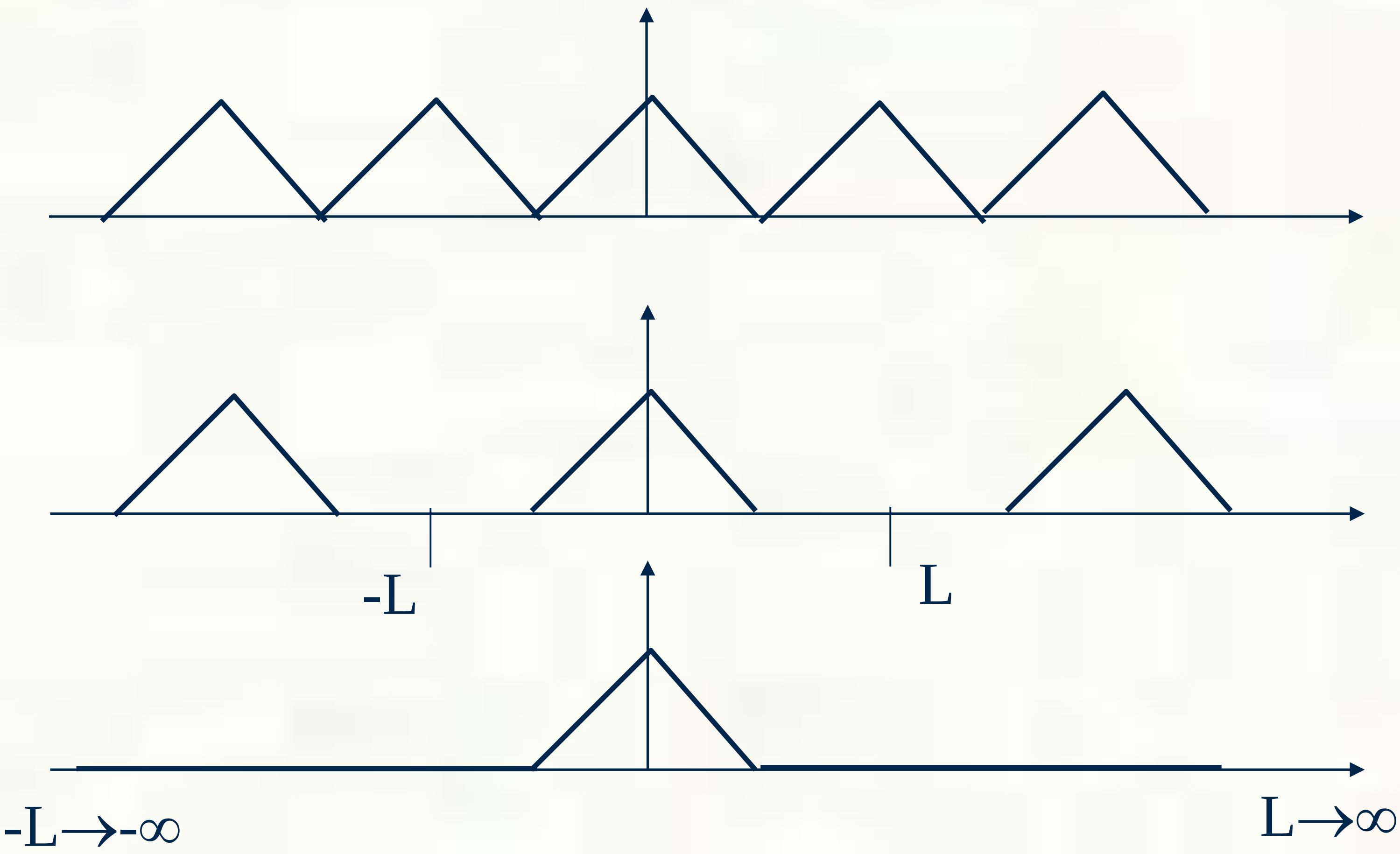
$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,e^{-in\omega_0 t}\,dt$$

$$= |F_n| e^{i\theta_n}$$

- So you could also write $f(t) = \sum_{n=-\infty}^{\infty} |F_n| e^{i(n\omega_0 t + \theta_n)}$

# Fourier Integrals

- For non-periodic applications (or a specialized Fourier series when the period of the function is infinite: $L \to \infty$)



$$f_L(x) \quad a_0 \quad {}^{\infty}_{n\ \ 1} \; a_n \cos \frac{n\pi x}{L} \quad b_n \sin \frac{n\pi x}{L}$$

$$a_0 \quad {}^{\infty}_{n\ \ 1} \; a_n \cos\ w_n x \quad b_n \sin\ w_n x \quad , w_n \quad \frac{n\pi}{L}$$

$${}^{\infty}_{n\ \ 0} \; a_n \cos\ w_n x \quad b_n \sin\ w_n x$$

$$\text{Note that}: \Delta w \quad w_{n\ \ 1} \quad w_n \quad \frac{(n \quad 1)\pi}{L} \quad \frac{n\pi}{L} \quad \frac{\pi}{L}$$

$$f_L(x)$$

$$\frac{1}{\pi} \; {}^{\infty}_{n\ \ 0} \; \cos\ w_n x \quad w \; {}^{L}_{L} \; f_L(v)\cos(w_n v)dv \quad \sin\ w_n x \quad w \; {}^{L}_{L} \; f_L(v)\sin(w_n v)dv$$

$$\text{As L} \to \infty, \quad \text{w} \to 0, \qquad (\,) \;\; w \to \quad (\,)dw$$

$$f_L(x)$$

$$\frac{1}{\pi}\;{}^{\infty}_{n\;\;0}\quad \cos\, w_n x \quad w \;{}^{L}_{L}\, f_L(v)\cos(w_n v)dv \quad \sin\, w_n x \quad w\;{}^{L}_{L}\, f_L(v)\sin(w_n v)dv$$

$$\frac{1}{\pi}\;{}^{\infty}_{0}\quad \cos\, wx \quad {}^{\infty}_{\infty}\, f(v)\cos(wv)dv \quad \sin\, wx \quad {}^{\infty}_{\infty}\, f(v)\sin(wv)dv\;\; dw$$

$${}^{\infty}_{0}\quad \frac{1}{\pi}\;{}^{\infty}_{\infty}\, f(v)\cos(wv)dv\,\cos(wx) \quad \frac{1}{\pi}\;{}^{\infty}_{\infty}\, f(v)\sin(wv)dv\,\sin(wx) \quad dw$$

$$f(x) \quad {}^{\infty}_{0}\quad A(w)\cos(wx) \quad B(w)\sin(wx)\, dw : \textbf{Fourier integral of f(x)}$$

$$\text{where } A(w) \quad \frac{1}{\pi}\;{}^{\infty}_{\infty}\, f(v)\cos(wv)dv,\, B(w) \quad \frac{1}{\pi}\;{}^{\infty}_{\infty}\, f(v)\sin(wv)dv$$

# Fourier Cosine & Sine Integrals

If the function f(x) is even $\quad A(w) \quad \dfrac{1}{\pi} {}_{\infty}^{\infty} f(v)\cos(wv)dv$

$$\frac{1}{\pi} {}_{\infty}^{0} (\ )dv \quad \frac{1}{\pi} {}_{0}^{\infty} (\ )dv \quad \frac{2}{\pi} {}_{0}^{\infty} f(v)\cos(wv)dv$$

$B(w) \quad 0$

$f(x) \quad {}_{0}^{\infty} A(w)\cos(wx)dw$ : Fourier Cosine Integral

If the function f(x) is odd $\quad B(w) \quad \dfrac{1}{\pi} {}_{\infty}^{\infty} f(v)\sin(wv)dv$

$A(w) \quad 0,\ f(x) \quad {}_{0}^{\infty} B(w)\sin(wx)dw$ : Fourier Sine Integral

# Example

$$\text{Let } f(x) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

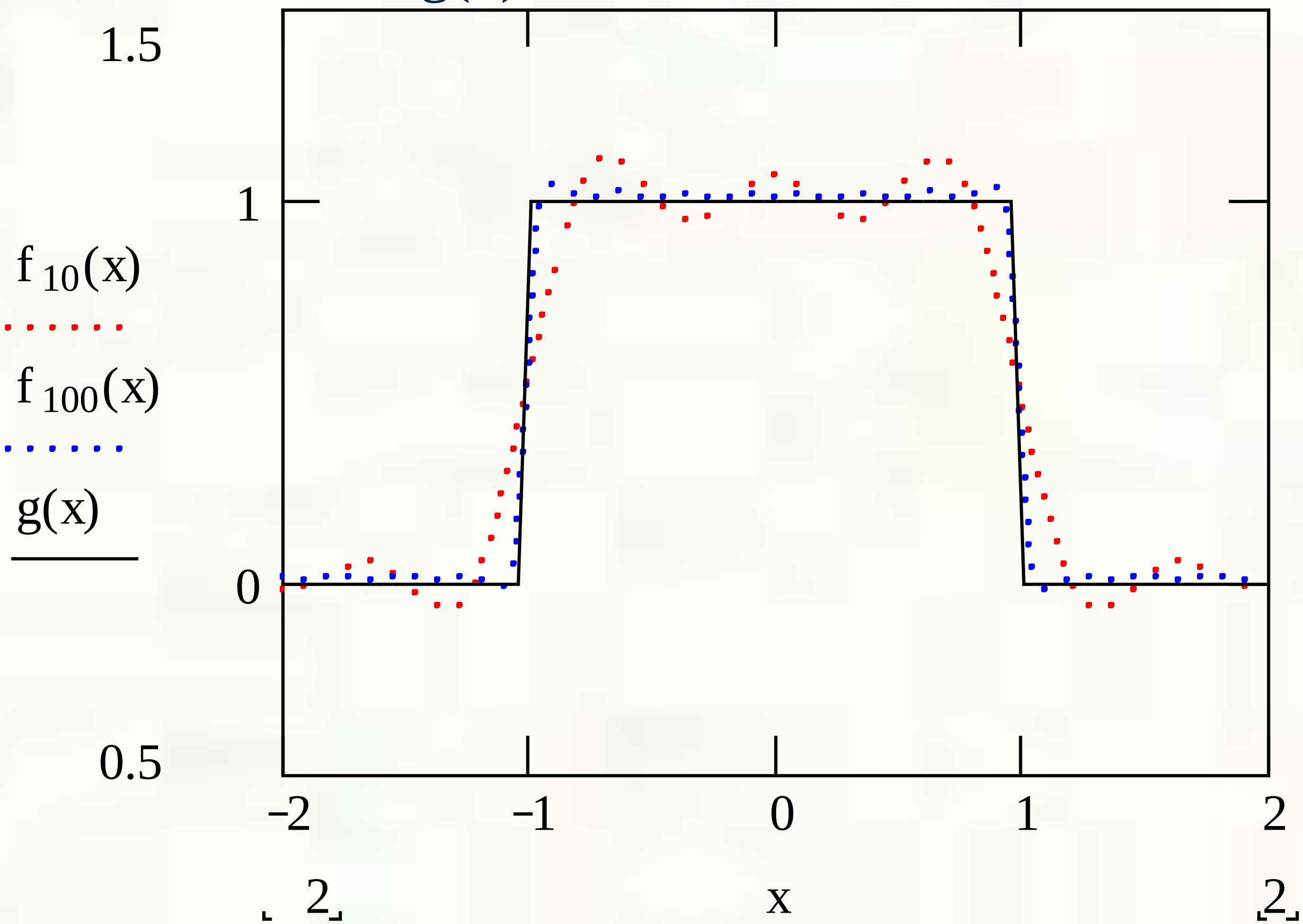
$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)\,dv = \frac{1}{\pi}\int_{-1}^{1}\cos(wv)\,dv = \frac{2\sin(w)}{w\pi}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} f(v)\sin(wv)\,dv = \frac{1}{\pi}\int_{-1}^{1}\sin(wv)\,dv = 0$$

The Fourier integral of f is

$$f(x) = \int_{0}^{\infty} A(w)\cos(wx)\,dw = \int_{0}^{\infty}\frac{2\sin(w)}{w\pi}\cos(wx)\,dw$$

$f_{10}$ integrate from 0 to 10
$f_{100}$ integrate from 0 to 100
g(x) the real function



Similar to Fourier series approximation, the Fourier integral approximation improves as the integration limit increases. It is expected that the integral will converges to the real function when the integration limit is increased to infinity.

Physical interpretation: The higher the integration limit means more higher frequency sinusoidal components have been included in the approximation. (similar effect has been observed when larger n is used in Fourier series approximation) This suggests that w can be interpreted as the frequency of each of the sinusoidal wave used to approximate the real function.

Suggestion: A(w) can be interpreted as the amplitude function of the specific sinusoidal wave. (similar to the Fourier coefficient in Fourier series expansion)

# Fourier Cosine Transform

For an even function f(x) :

$$f(x) = \int_0^\infty A(w)\cos(wx)\,dw, \quad \text{where } A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos(wv)\,dv.$$

Define $A(w) = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$

$$\hat{f}_c(w) = \sqrt{\frac{\pi}{2}}A(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(wx)\,dx, \quad v \text{ has been replaced by x}$$

$\hat{f}_c(w)$ is called the Fourier cosine transform of f(x)

$$f(x) = \int_0^\infty A(w)\cos(wx)\,dw = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos(wx)\,dw$$

$f(x)$ is the inverse Fourier cosine transform of $\hat{f}_c(w)$

# Fourier Sine Transform

Similarly, for an odd function f(x) :

$$f(x) = \int_0^\infty B(w)\sin(wx)\,dw, \quad \text{where } B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin(wv)\,dv.$$

Define $B(w) = \sqrt{\frac{2}{\pi}}\,\hat{f}_S(w)$

$$\hat{f}_S(w) = \sqrt{\frac{\pi}{2}}B(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(wx)\,dx, \quad v \text{ has been replaced by x}$$

$\hat{f}_S(w)$ is called the Fourier sine transform of f(x)

$$f(x) = \int_0^\infty B(w)\sin(wx)\,dw = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_S(w)\sin(wx)\,dw$$

$f(x)$ is the inverse Fourier sine transform of $\hat{f}_S(w)$

# Fourier transform

- We now have $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}$

$$F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-in\omega_0 t}\, dt$$

- Let's not use just discrete frequencies, $n\omega_0$ , we'll allow them to vary continuously too
- We'll get there by setting $t_0 = -T/2$ and taking limits as $T$ and $n$ approach $\infty$

# Fourier transform

$$f(t) \quad {}_{n \quad \infty}^{\infty} F_n e^{in\omega_0 t} \quad {}_{n \quad \infty}^{\infty} e^{in\omega_0 t} \frac{1}{T} {}_{T/2}^{T/2} f(t) e^{\ in\omega_0 t} dt$$

$$\quad {}_{n \quad \infty}^{\infty} e^{in\frac{2\pi}{T}t} \frac{2\pi}{T} \frac{1}{2\pi} {}_{T/2}^{T/2} f(t) e^{\ in\frac{2\pi}{T}t} dt$$

$$\lim_{T\to\infty} \frac{2\pi}{T} \quad d\omega \qquad \lim_{n\to\infty} n\, d\omega \quad \omega$$

$$f(t) \quad {}_{\infty}^{\infty} e^{i\omega t} d\omega \frac{1}{2\pi} {}_{\infty}^{\infty} f(t) e^{\ i\omega t} dt$$

$$\frac{1}{\sqrt{2\pi}} {}_{\infty}^{\infty} e^{i\omega t} \ \frac{1}{\sqrt{2\pi}} {}_{\infty}^{\infty} f(t) e^{\ i\omega t} dt \ \ d\omega$$

$$\frac{1}{\sqrt{2\pi}} {}_{\infty}^{\infty} e^{i\omega t} F(\omega) d\omega$$

# Fourier transform

- So we have (unitary form, angular frequency)

$$\mathcal{F}(f(t)) = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\mathcal{F}^{-1}(F(\omega)) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

- Alternatives (Laplace form, angular frequency)

$$\mathcal{F}(f(t)) = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\mathcal{F}^{-1}(F(\omega)) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

# Fourier transform

- Ordinary frequency

$$\frac{\omega}{2\pi}$$

$$\mathsf{F}(f(t)) \quad F(\quad) \quad {}_{\infty}^{\infty} \; f(t)e^{i \quad t}dt$$

$$\mathsf{F}^{-1}(F(\quad)) \quad f(t) \quad {}_{\infty}^{\infty} \; F(\quad)e^{i \quad t}d$$

# Fourier transform

- Some sufficient conditions for application
  - Dirichlet conditions
    - $${}_{\infty}^{\infty} \left| f(t) \right| dt \quad \infty$$
    - *f(t)* has finite maxima and minima within any finite interval
    - *f(t)* has finite number of discontinuities within any finite interval
  - Square integrable functions ($L^2$ space)

$${}_{\infty}^{\infty} [f(t)]^2 dt \quad \infty$$

  - Tempered distributions, like Dirac delta

$$\mathsf{F}( \quad (t)) \quad \frac{1}{\sqrt{2\pi}}$$

# Fourier transform

- Complex form – orthonormal basis functions for space of tempered distributions

$$\int_{\infty}^{\infty} \frac{e^{i\omega_1 t}}{\sqrt{2\pi}} \frac{e^{\;i\omega_2 t}}{\sqrt{2\pi}}\, dt \qquad (\omega_1 \quad \omega_2)$$

# Convolution theorem

**Theorem**

$$\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(fg) = \mathcal{F}(f)*\mathcal{F}(g)$$

$$\mathcal{F}^{-1}(F*G) = \mathcal{F}^{-1}(F)\mathcal{F}^{-1}(G)$$

$$\mathcal{F}^{-1}(FG) = \mathcal{F}^{-1}(F)*\mathcal{F}^{-1}(G)$$

**Proof (1)**

$$\mathcal{F}(f*g) = \int_{-\infty}^{\infty} f(t')g(t-t') \int_{-\infty}^{\infty} e^{-i\omega t}\,dt\,dt'$$

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt' \int_{-\infty}^{\infty} g(t-t')e^{-i\omega(t-t')}dt$$

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt' \int_{-\infty}^{\infty} g(t'')e^{-i\omega t''}dt''$$

$$= \mathcal{F}(f)\mathcal{F}(g)$$

# THANK YOU