Eigenvalues and Eigenvectors

1 Eigen values and Eigenvectors

2 Diagonalization

3 Symmetric Matrices and Orthogonal Diagonalization

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1 Eigenvalues and Eigenvectors

- Eigenvalue problem (one of the most important problems in the linear algebra):

  If A is an $n\times n$ matrix, do there exist nonzero vectors $\mathbf{x}$ in R^n such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

  (The term eigenvalue is from the German word *Eigenwert*, meaning "proper value")

- Eigenvalue and Eigenvector :

  A: an $n\times n$ matrix

  λ: a scalar (could be **zero**)

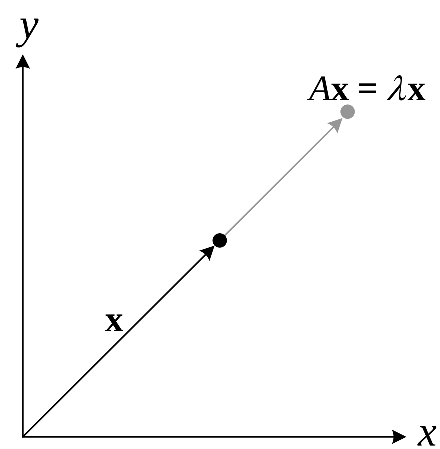
  $\mathbf{x}$: a **nonzero** vector in R^n

Eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}$$

Eigenvector

※ Geometric Interpretation



- Ex 1: Verifying eigenvalues and eigenvectors

$$A=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\quad \mathbf{x}_1=\begin{bmatrix}1\\0\end{bmatrix}\quad \mathbf{x}_2=\begin{bmatrix}0\\1\end{bmatrix}$$

Eigenvalue

$$A\mathbf{x}_1=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}=2\mathbf{x}_1$$

Eigenvector

Eigenvalue

$$A\mathbf{x}_2=\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}=-1\begin{bmatrix}0\\1\end{bmatrix}=(-1)\mathbf{x}_2$$

Eigenvector

- Thm. 1: The eigenspace corresponding to λ of matrix A

  If A is an $n\times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ **together with the zero vector** is a subspace of R^n. This subspace is called the eigenspace of λ

Proof:

$\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to λ

(i.e., $A\mathbf{x}_1=\lambda\mathbf{x}_1,\ \ A\mathbf{x}_2=\lambda\mathbf{x}_2$)

(1) $A(\mathbf{x}_1+\mathbf{x}_2)=A\mathbf{x}_1+A\mathbf{x}_2=\lambda\mathbf{x}_1+\lambda\mathbf{x}_2=\lambda(\mathbf{x}_1+\mathbf{x}_2)$

(i.e., $\mathbf{x}_1+\mathbf{x}_2$ is also an eigenvector corresponding to λ)

(2) $A(c\mathbf{x}_1)=c(A\mathbf{x}_1)=c(\lambda\mathbf{x}_1)=\lambda(c\mathbf{x}_1)$

(i.e., $c\mathbf{x}_1$ is also an eigenvector corresponding to λ)

Since this set is closed under vector addition and scalar multiplication, this set is a subspace of R^n .

- Ex 3: Examples of eigenspaces on the *xy*-plane

For the matrix A as follows, the corresponding eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

For the eigenvalue $\lambda_1 = -1$, corresponding vectors are any vectors on the x-axis

$$A\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

※ Thus, the eigenspace corresponding to $\lambda = -1$ is the x-axis, which is a subspace of R^2

For the eigenvalue $\lambda_2 = 1$, corresponding vectors are any vectors on the y-axis

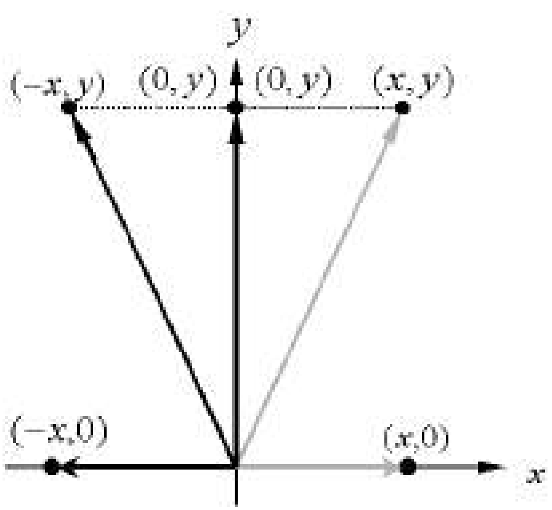
$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

※ Thus, the eigenspace corresponding to $\lambda = 1$ is the y-axis, which is a subspace of R^2

※ Geometrically speaking, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection to the y-axis, i.e., left multiplying A to $\mathbf{v}$ can transform $\mathbf{v}$ to another vector in the same vector space

$$A\mathbf{v} = A\begin{bmatrix} x \\ y \end{bmatrix} = A\left(\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ 0 \end{bmatrix} + A\begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$= -1\begin{bmatrix} x \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



- Thm. 2: Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$

  Let A be an $n\times n$ matrix.

  (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$

  (2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$

- Note: follwing the definition of the eigenvalue problem

  $$A\mathbf{x} = \lambda\mathbf{x} \;\Rightarrow\; A\mathbf{x} = \lambda I\mathbf{x} \;\Rightarrow\; (\lambda I - A)\mathbf{x} = \mathbf{0}$$ (homogeneous system)

  $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nonzero solutions for $\mathbf{x}$ iff $\det(\lambda I - A) = 0$

  (The above iff results comes from the equivalent conditions on Slide 4.101)

- Characteristic equation of A:

  $$\det(\lambda I - A) = 0$$

- Characteristic polynomial of $A \in M_{n\times n}$:

  $$\det(\lambda I - A) = \left|(\lambda I - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Ex 4: Finding eigenvalues and eigenvectors

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalue: $\lambda_1 = -1, \lambda_2 = -2$

$$(1)\ \lambda_1 = -1 \ \Rightarrow (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2)\ \lambda_2 = -2 \ \Rightarrow (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3s \\ s \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad s \neq 0$$

- Ex 5: Finding eigenvalues and eigenvectors

Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of $\lambda = 2$:

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \;\; s, t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| \, s, t \in R \right\} : \text{the eigenspace of } A \text{ corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2

- Notes:

    (1) If an eigenvalue λ_1 occurs as a multiple root (k times) for the characteristic polynominal, then λ_1 has multiplicity k

    (2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace. (In Ex. 5, k is 3 and the dimension of its eigenspace is 2)

- Ex 6：Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

Eigenvalues: $\lambda_1=1, \lambda_2=2, \lambda_3=3$

$$(1)\ \lambda_1 = 1 \quad \Rightarrow (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{G.-J.E.}}{\Rightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$

※The dimension of the eigenspace of $\lambda_1 = 1$ is 2

(2) $\lambda_2 = 2 \;\Rightarrow (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\overset{\text{G.-J.E.}}{\Longrightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_2 = 2$

※The dimension of the eigenspace of $\lambda_2 = 2$ is 1

(3) $\lambda_3 = 3 \Rightarrow (\lambda_3 I - A)\mathbf{x} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\overset{\text{G.-J.E.}}{\Longrightarrow} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_3 = 3$

※The dimension of the eigenspace of $\lambda_3 = 3$ is 1

- Thm. 3: Eigenvalues for triangular matrices

  If A is an $n\times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal

- Ex 7: Finding eigenvalues for triangular and diagonal matrices

$$\text{(a) } A=\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \qquad \text{(b) } A=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$\text{(a) } |\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ -5 & -3 & \lambda+3 \end{vmatrix} = (\lambda-2)(\lambda-1)(\lambda+3)=0$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -3$$

$$\text{(b) } \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 0,\ \lambda_4 = -4,\ \lambda_5 = 3$$

- Ex 8: Finding eigenvalues and eigenvectors for standard matrices

Find the eigenvalues and corresponding eigenvectors for

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

※ A is the standard matrix for $T(x_1, x_2, x_3) = (x_1 + 3x_2, 3x_1 + x_2, -2x_3)$ (see Slides 7.19 and 7.20)

Sol:

$$|\lambda I - A| = \begin{bmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{bmatrix} = (\lambda+2)^2(\lambda-4)=0$$

$$\Rightarrow \text{eigenvalues } \lambda_1 = 4,\ \lambda_2 = -2$$

For $\lambda_1 = 4$, the corresponding eigenvector is $(1,\ 1,\ 0)$.

For $\lambda_2 = -2$, the corresponding eigenvectors are $(1,\ -1,\ 0)$ and $(0,\ 0,\ 1)$.

▪ Transformation matrix A' for nonstandard bases

Suppose B is the standard basis of R^n. Since the coordinate matrix of a vector relative to the standard basis consists of the components of that vector, i.e., for any $\mathbf{x}$ in R^n, $\mathbf{x} = [\mathbf{x}]_B$.

$$T(\mathbf{x}) = A\mathbf{x} \Rightarrow \left[T(\mathbf{x})\right]_B = A\left[\mathbf{x}\right]_B, \text{ where } A = \left[\left[T(\mathbf{e}_1)\right]_B \ \left[T(\mathbf{e}_2)\right]_B \cdots \left[T(\mathbf{e}_n)\right]_B\right]$$

is the standard matrix for T or the matrix of T relative to the standard basis B

The above theorem can be extended to consider a nonstandard basis B', which consists of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$$\left[T(\mathbf{x})\right]_{B'} = A'\left[\mathbf{x}\right]_{B'}, \text{ where } A' = \left[\left[T(\mathbf{v}_1)\right]_{B'} \ \left[T(\mathbf{v}_2)\right]_{B'} \cdots \left[T(\mathbf{v}_n)\right]_{B'}\right]$$

is the transformation matrix for T relative to the basis B'

2 Diagonalization

- Diagonalization problem :

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix :

  Definition 1: A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (i.e., P diagonalizes A)

  Definition 2: A square matrix A is called **diagonalizable** if A is **similar** to a diagonal matrix

  ※ In Sec. 6.4, two square matrices A and B are **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$.

- Notes:

  This section shows that the eigenvalue and eigenvector problem is closely related to the diagonalization problem

- Thm. 4: Similar matrices have the same eigenvalues

If A and B are similar $n\times n$ matrices, then they have the same eigenvalues

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

For any diagonal matrix in the form of $D = \lambda I$, $P^{-1}DP = D$

Consider the characteristic equation of B:

$$\begin{aligned}|\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda IP - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\ &= |\lambda I - A|\end{aligned}$$

Since A and B have the same characteristic equation, they are with the same eigenvalues

※ Note that the eigenvectors of A and B are not necessarily identical

- Ex 1: Eigenvalue problems and diagonalization programs

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{The eigenvalues}: \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\ \lambda=4 \Rightarrow \text{ the eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

(2) $\lambda = -2 \Rightarrow$ the eigenvector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Note: If $P = [\mathbf{p}_2 \quad \mathbf{p}_1 \quad \mathbf{p}_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Thm. 5: Condition for diagonalization

  An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors

※ If there are n linearly independent eigenvectors, it does not imply that there are n distinct eigenvalues. In an extreme case, it is possible to have only one eigenvalue with the multiplicity n, and there are n linearly independent eigenvectors for this eigenvalue

※ On the other hand, if there are n distinct eigenvalues, then there are n linearly independent eigenvectors, and thus A must be diagonalizable

- Ex 4: A matrix that is not diagonalizable

Show that the following matrix is not diagonalizable

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

The eigenvalue $\lambda_1=1,$ and then solve $(\lambda_1 I-A)\mathbf{x}=\mathbf{0}$ for eigenvectors

$$\lambda_1 I-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\Rightarrow \text{eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$$

Since A does not have two linearly independent eigenvectors,

A is not diagonalizable

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \cdots \mathbf{p}_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Step 2: Let $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \quad i = 1, \ 2, \ldots, \ n$

- Ex 5: Diagonalizing a matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

$$\text{The eigenvalues}: \ \lambda_1 = 2,\ \lambda_2 = -2,\ \lambda_3 = 3$$

$$\lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2 \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3 \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \quad \Rightarrow \quad \text{eigenvector } \mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \text{ and it follows that}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Note: a quick way to calculate A^k based on the diagonalization technique

$$(1)\ D=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} \Rightarrow D^k=\begin{bmatrix}\lambda_1^k & 0 & \cdots & 0\\ 0 & \lambda_2^k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n^k\end{bmatrix}$$

$$(2)\ D=P^{-1}AP \Rightarrow D^k=\underbrace{P^{-1}AP\ P^{-1}AP\cdots P^{-1}AP}_{\text{repeat } k \text{ times}}=P^{-1}A^kP$$

$$A^k=PD^kP^{-1},\ \text{where } D^k=\begin{bmatrix}\lambda_1^k & 0 & \cdots & 0\\ 0 & \lambda_2^k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n^k\end{bmatrix}$$

- Thm. 6: Sufficient conditions for diagonalization

    If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and thus A is diagonalizable.

- Ex 7: Determining whether a matrix is diagonalizable

$$A=\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol: Because A is a triangular matrix, its eigenvalues are

$$\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = -3$$

According to Thm. 6, because these three values are distinct, A is diagonalizable

- Ex 8: Finding a diagonalized matrix for a linear transformation

Let $T: R^3 \rightarrow R^3$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

Find a basis B' for R^3 such that the matrix for T relative to B' is diagonal

Sol:

The standard matrix for T is given by

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From Ex. 5 you know that $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 3$ and thus A is diagonalizable.

$$B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(-1,\ 0,\ 1), (1,\ -1,\ 4), (-1,\ 1,\ 1)\}$$

The matrix for T relative to this basis is

$$A' = \left[[T(\mathbf{v}_1)]_{B'} \quad [T(\mathbf{v}_2)]_{B'} \quad [T(\mathbf{v}_3)]_{B'} \right]$$
$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3 Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix :

A square matrix A is symmetric if it is equal to its transpose:

$$A = A^T$$

- Ex 1: Symmetric matrices and nonsymetric matrices

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$ (symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$ (symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$ (nonsymmetric)

- Thm 7: Eigenvalues of symmetric matrices

If A is an $n \times n$ "symmetric" matrix, then the following properties are true

(1) A is diagonalizable (symmetric matrices (except the matrices in the form of $A = aI$, in which case A is already diagonal) are guaranteed to have n linearly independent eigenvectors and thus be diagonalizable)

(2) All eigenvalues of A are real numbers

(3) If λ is an eigenvalue of A with the multiplicity to be k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k

※ The above theorem is called the **Real Spectral Theorem**, and the set of eigenvalues of A is called the **spectrum** of A

- Ex 2:

Prove that a 2 × 2 symmetric matrix is diagonalizable

$$A=\begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Pf: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

As a function in λ, this quadratic polynomial function has a nonnegative discriminant as follows

$$\begin{aligned}(a+b)^2 - 4(1)(ab-c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= \underline{(a-b)^2 + 4c^2} \geq 0 \Rightarrow \text{real-number solutions}\end{aligned}$$

(1) $(a-b)^2+4c^2=0$

$$\Rightarrow a=b,\ c=0$$

$$A=\begin{bmatrix} a & c \\ c & b \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$ itself is a diagonal matrix

※ Note that in this case, A has one eigenvalue, a, whose multiplicity is 2, and the two eigenvectors are linearly independent

(2) $(a-b)^2+4c^2>0$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. According to Thm. 6, A is diagonalizable

- Orthogonal matrix :

  A square matrix P is called orthogonal if it is invertible and

$$P^{-1} = P^T \text{ (or } PP^T = P^T P = I\text{)}$$

- Thm. 8: Properties of orthogonal matrices

  An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set

Pf: Suppose the column vectors of P form an orthonormal set, i.e.,

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}, \text{ where } \mathbf{p}_i \cdot \mathbf{p}_j = 0 \text{ for } i \neq j \text{ and } \mathbf{p}_i \cdot \mathbf{p}_i = 1$$

$$P^T P = \begin{bmatrix} \mathbf{p}_1^T \mathbf{p}_1 & \mathbf{p}_1^T \mathbf{p}_2 & \cdots & \mathbf{p}_1^T \mathbf{p}_n \\ \mathbf{p}_2^T \mathbf{p}_1 & \mathbf{p}_2^T \mathbf{p}_2 & \cdots & \mathbf{p}_2^T \mathbf{p}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_n^T \mathbf{p}_1 & \mathbf{p}_n^T \mathbf{p}_2 & \cdots & \mathbf{p}_n^T \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \cdot \mathbf{p}_1 & \mathbf{p}_1 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_1 \cdot \mathbf{p}_n \\ \mathbf{p}_2 \cdot \mathbf{p}_1 & \mathbf{p}_2 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_2 \cdot \mathbf{p}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p}_n \cdot \mathbf{p}_1 & \mathbf{p}_n \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_n \cdot \mathbf{p}_n \end{bmatrix} = I_n$$

It implies that $P^{-1} = P^T$ and thus P is orthogonal

- Ex 5: Show that P is an orthogonal matrix.

$$P=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Sol: If P is a orthogonal matrix, then $P^{-1}=P^{T} \Rightarrow PP^{T}=I$

$$PP^{T}=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

Moreover, let $\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{-2}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3\sqrt{5}} \end{bmatrix}$, and $\mathbf{p}_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$,

we can produce $\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$ and $\mathbf{p}_1 \cdot \mathbf{p}_1 = \mathbf{p}_2 \cdot \mathbf{p}_2 = \mathbf{p}_3 \cdot \mathbf{p}_3 = 1$

So, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set.

- Thm. 9: Properties of symmetric matrices

  Let A be an $n \times n$ "symmetric" matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Pf:

$$\lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) = (\lambda_1\mathbf{x}_1) \cdot \mathbf{x}_2 = (A\mathbf{x}_1) \cdot \mathbf{x}_2 = (A\mathbf{x}_1)^T \mathbf{x}_2 = (\mathbf{x}_1^T A^T)\mathbf{x}_2$$

because A is symmetric

$$= (\mathbf{x}_1^T A)\mathbf{x}_2 = \mathbf{x}_1^T(A\mathbf{x}_2) = \mathbf{x}_1^T(\lambda_2\mathbf{x}_2) = \mathbf{x}_1 \cdot (\lambda_2\mathbf{x}_2) = \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2)$$

The above equation implies $(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0,$ and because $\lambda_1 \neq \lambda_2,$ it follows that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal

※ For distinct eigenvalues of a symmetric matrix, their corresponding eigenvectors are orthogonal and thus linearly independent to each other

※ Note that there may be multiple $\mathbf{x}_1$'s and $\mathbf{x}_2$'s corresponding to λ_1 and λ_2

- Orthogonal diagonalization :

  A matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal

- Thm. 10: Fundamental theorem of symmetric matrices

  An $n\times n$ matrix A is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric

Pf:

$(\Rightarrow)$

A is orthogonally diagonalizable

$\Rightarrow D = P^{-1}AP$ is diagonal, and P is an orthogonal matrix s.t. $P^{-1} = P^T$

$$\Rightarrow A = PDP^{-1} = PDP^T \Rightarrow A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

Orthogonal diagonalization of a symmetric matrix:

Let A be an $n \times n$ symmetric matrix.

(1) Find all eigenvalues of A and determine the multiplicity of each

※ According to Thm. 9, eigenvectors corresponding to distinct eigenvalues are orthognoal

(2) For each eigenvalue of multiplicity 1, choose the unit eigenvector

(3) For each eigenvalue of the multiplicity to be $k \geq 2$, find a set of k linearly independent eigenvectors. If this set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is not orthonormal, apply the Gram-Schmidt orthonormalization process

It is known that G.-S. process is a kind of linear transformation, i.e., the produced vectors can be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ (see Slide 5.55),

i. Since $A\mathbf{v}_1 = \lambda\mathbf{v}_1, A\mathbf{v}_2 = \lambda\mathbf{v}_2, \ldots, A\mathbf{v}_k = \lambda\mathbf{v}_k$,

$\Rightarrow A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = \lambda(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k)$

$\Rightarrow$ The produced vectors through the G.-S. process are still eigenvectors for λ

ii. Since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are orthogonal to eigenvectors corresponding to other different eigenvalues (according to Thm. 7.9), $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is also orthogonal to eigenvectors corresponding to other different eigenvalues.

(4) The composite of steps (2) and (3) produces an orthonormal set of n eigenvectors. Use these orthonormal and thus linearly independent eigenvectors as column vectors to form the matrix P.

i. According to Thm. 8, the matrix P is orthogonal

ii. Following the diagonalization process , $D = P^{-1}AP$ is diagonal

Therefore, the matrix A is orthogonally diagonalizable

- Ex 7: Determining whether a matrix is orthogonally diagonalizable

	Symmetric matrix	Orthogonally diagonalizable
$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	○	○
$A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$	×	×
$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	×	×
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$	○	○

- Ex 9: Orthogonal diagonalization

Find an orthogonal matrix P that diagonalizes A.

$$A=\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

Sol:

(1) $|\lambda I - A| = (\lambda-3)^2(\lambda+6)=0$

$\lambda_1=-6,\ \lambda_2=3$ (has a multiplicity of 2)

(2) $\lambda_1=-6,\ \mathbf{v}_1=(1,\ -2,\ 2) \ \Rightarrow\ \mathbf{u}_1=\dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}=(\frac{1}{3},\ \frac{-2}{3},\ \frac{2}{3})$

(3) $\lambda_2=3,\ \mathbf{v}_2=(2,\ 1,\ 0),\ \mathbf{v}_3=(-2,\ 4,\ 5)$

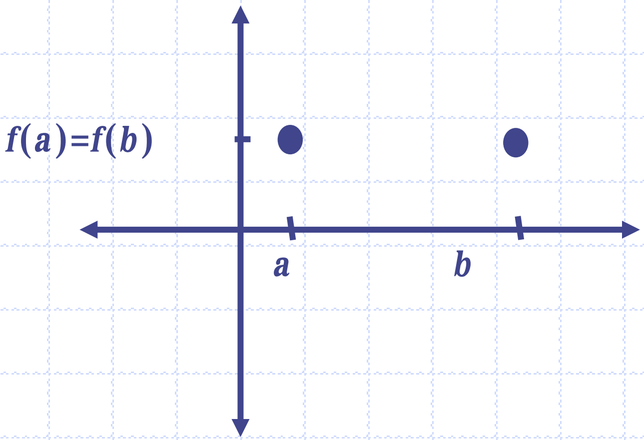
orthogonal

Rolle’s Theorem and the Mean Value Theorem

Rolle's Theorem

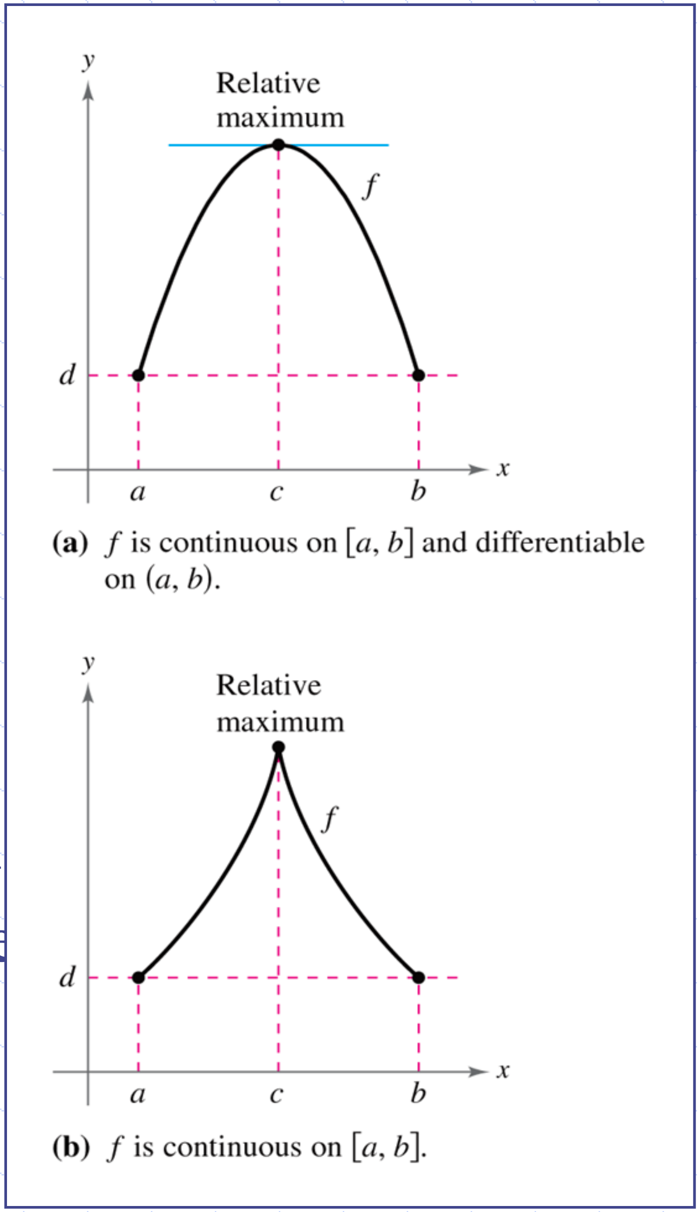
If you connect from $f\,(a)$ to $f\,(b)$ with a smooth curve,

there will be at least one place where $f\,'(c) = 0$



Rolle's Theorem

Rolle's theorem is an important basic result about differentiable functions. Like many basic results in the calculus it seems very obvious. It just says that between any two points where the graph of the differentiable function $f(x)$ cuts the horizontal line there must be a point where $f'(x) = 0$. The following picture illustrates the theorem.



Rolle's Theorem

If two points at the same <u>height</u> are connected by a continuous, differentiable function, then there has to be <u>at least one</u> place between those two points where the derivative, or slope, is <u>zero</u>.

Rolle's Theorem

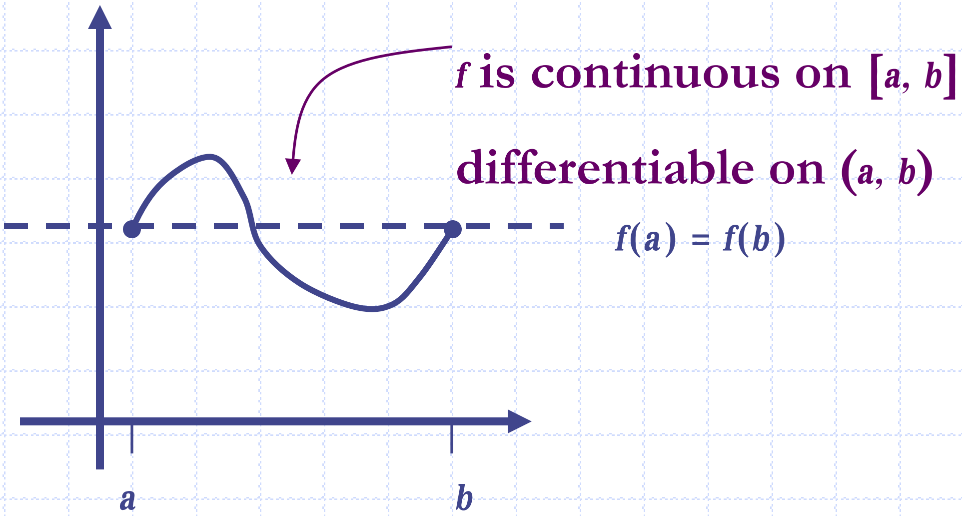
If 1) $f(x)$ is continuous on $[a, b]$,

2) $f(x)$ is differentiable on (a, b), and

3) $f(a) = f(b)$

then there is at least one value of x on (a, b), call it c, such that

$f'(c) = 0$.



Example

Example 1 $f(x) = x^4 - 2x^2 \quad \text{on}\,[-2, 2]$

(f is continuous and differentiable)

$$f(-2) = 8 = f(2)$$

Since , then Rolle's Theorem applies…

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0$$

then, $x = -1$, $x = 0$, and $x = 1$

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of c.

Example 2 $f(x) = 4 - x^2$ $[-2, 2]$

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 3 $f(x) = x^3 - x$ $[-1, 1]$

Example

Example 4 $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ on $[-1, 1]$

(Graph the function over the interval on your calculator)

continuous on [-1, 1]

not differentiable at 0

not differentiable on (-1, 1)

$f(-1) = 1 = f(1)$

Rolle's Theorem Does NOT apply since

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 5 $$f(x) = \frac{x^2 + 4}{x^2} \qquad [-2, 2]$$

Note

When working with Rolle's make sure you

1. State $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).
2. Show that $f(a) = f(b)$.
3. State that there exists at least one $x = c$ in (a, b) such that $f'(c) = 0$.

This theorem only guarantees the existence of an extrema in an open interval. It does not tell you how to find them or how many to expect. If YOU can not find such extrema, it does not mean that it can not be found. In most of cases, it is enough to know the existence of such extrema.

Mean Value Theorem- MVT

The Mean Value Theorem is one of the most important theoretical tools in Calculus. It states that if $f(x)$ is defined and continuous on the interval $[a,b]$ and differentiable on (a,b), then there is at least one number c in the interval (a,b) (that is $a<c<b$) such that

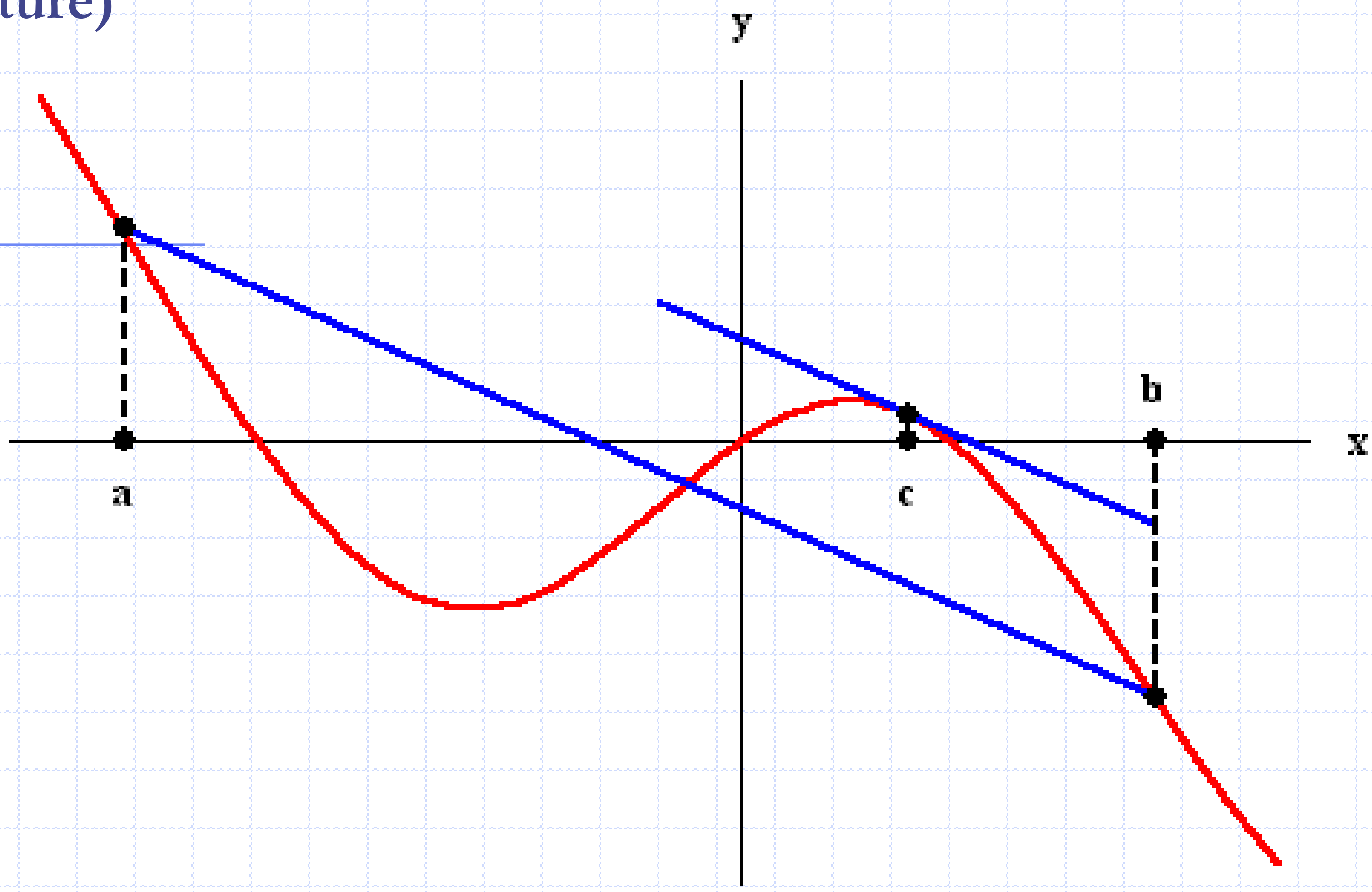
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, there exists a point in the interval (a,b) which has a horizontal tangent. In fact, the Mean Value Theorem can be stated also in terms of slopes. Indeed, the number

$$\frac{f(b) - f(a)}{b - a}$$

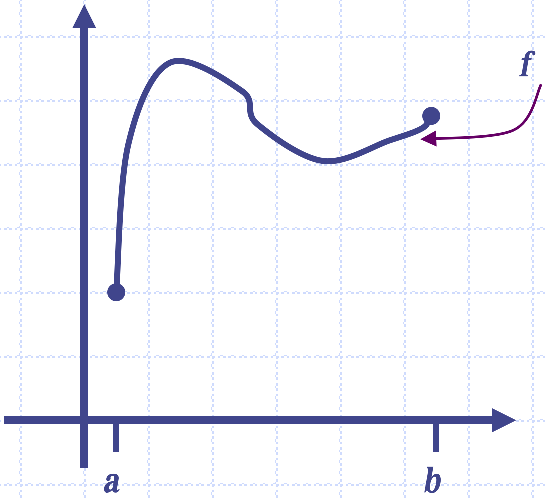
is the slope of the line passing through $(a, f(a))$ and $(b, f(b))$. So the conclusion of the Mean Value Theorem states that there exists a point such that the tangent line is parallel to the line passing through $(a, f(a))$ and $(b, f(b))$.

(see Picture)



The special case, when $f(a) = f(b)$ is known as Rolle's Theorem.
In this case, we have $f'(c) = 0$.

Mean Value Theorem- MVT



If: *f* **is continuous on [*a*, *b*],**

differentiable on (*a*, *b*)

Then: there is a *c* in (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example

Example 6 $f(x) = x^3 - x^2 - 2x$ on [-1,1]

(f is continuous and differentiable)

MVT applies

$$f'(x) = 3x^2 - 2x - 2$$

$$f'(c) = \frac{-2-0}{1-(-1)} = -1$$

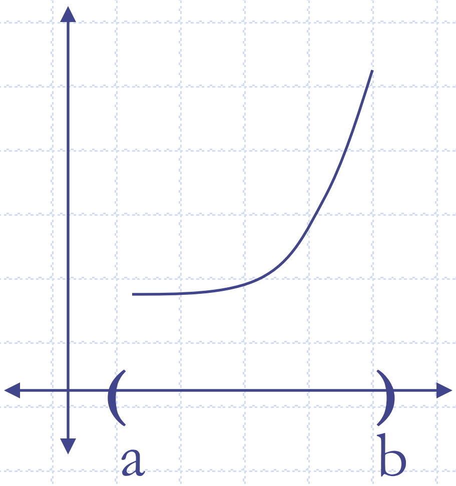
$$3c^2 - 2c - 2 = -1$$

$$(3c+1)(c-1) = 0$$

$$c = -\frac{1}{3}, \quad c = 1$$

Mean Value Theorem- MVT

Note:



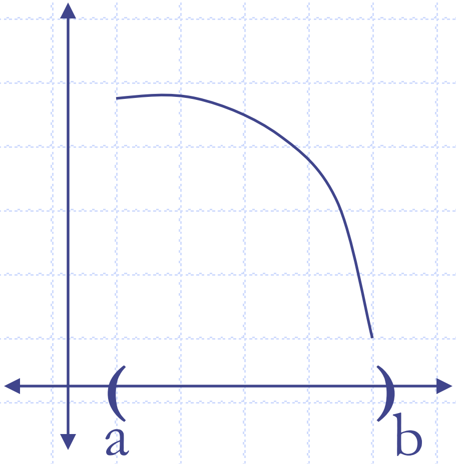
$$f'(x) > 0 \text{ on } (a,b) \quad \Rightarrow$$

$$f \text{ is increasing on } (a,b)$$

The graph of f is rising

Mean Value Theorem- MVT

Note:



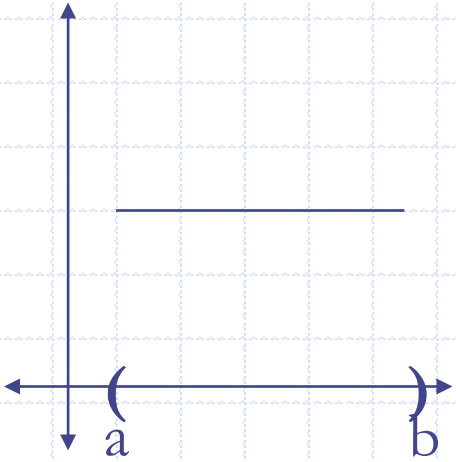
$$f'(x) < 0 \text{ on } (a,b) \quad \Rightarrow$$

$$f \text{ is decreasing on } (a,b)$$

The graph of f is falling

Mean Value Theorem- MVT

Note:



f is constant on (a,b)

The graph of f is level

Example

Example 7

$$f(x) = x^2 - 6x + 12$$

$$\begin{aligned} f'(x) &= 2x - 6 \\ &= 2(x-3) \\ &= 0 \text{ iff } x = 3 \end{aligned}$$

Finding a Tangent Line

Example 8 **Find all values of c in the open interval (a, b) such that**

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

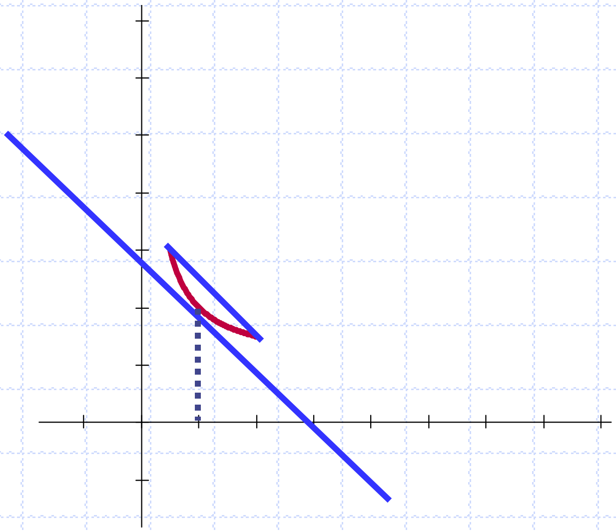
$$f(x) = \frac{x+1}{x}, \ [\tfrac{1}{2}, 2]$$

$$\frac{f(2) - f(1/2)}{2 - 1/2} = \frac{3/2 - 3}{3/2} = -1$$

$$f'(x) = \frac{d}{dx}\left(1 + \frac{1}{x}\right) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2} = -1$$

$c = 1$



Application of MVT

Example 9 When an object is removed from a furnace and placed in an environment with a constant temperature of 90° F, its core temperature is 1500° F. Five hours later the core temperature is 390° F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222° F per hour.

Solution

Let $g(t)$ be the temperature of the object.

Then $g(0) = 1500$, $g(5) = 390$

$$\text{Avg. Temp.} = \frac{g(5) - g(0)}{5 - 0} = \frac{390 - 1500}{5} = -222$$

By MVT, there exists a time $0 < t_o < 5$, such that $g'(t_o) = -222^{\circ}$ F

Application of MVT

Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a truck passes the first patrol car, its speed is clocked at 55 mph. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 mph. Prove that the truck must have exceeded the speed limit of 55 mph at some time during the 4 minutes.

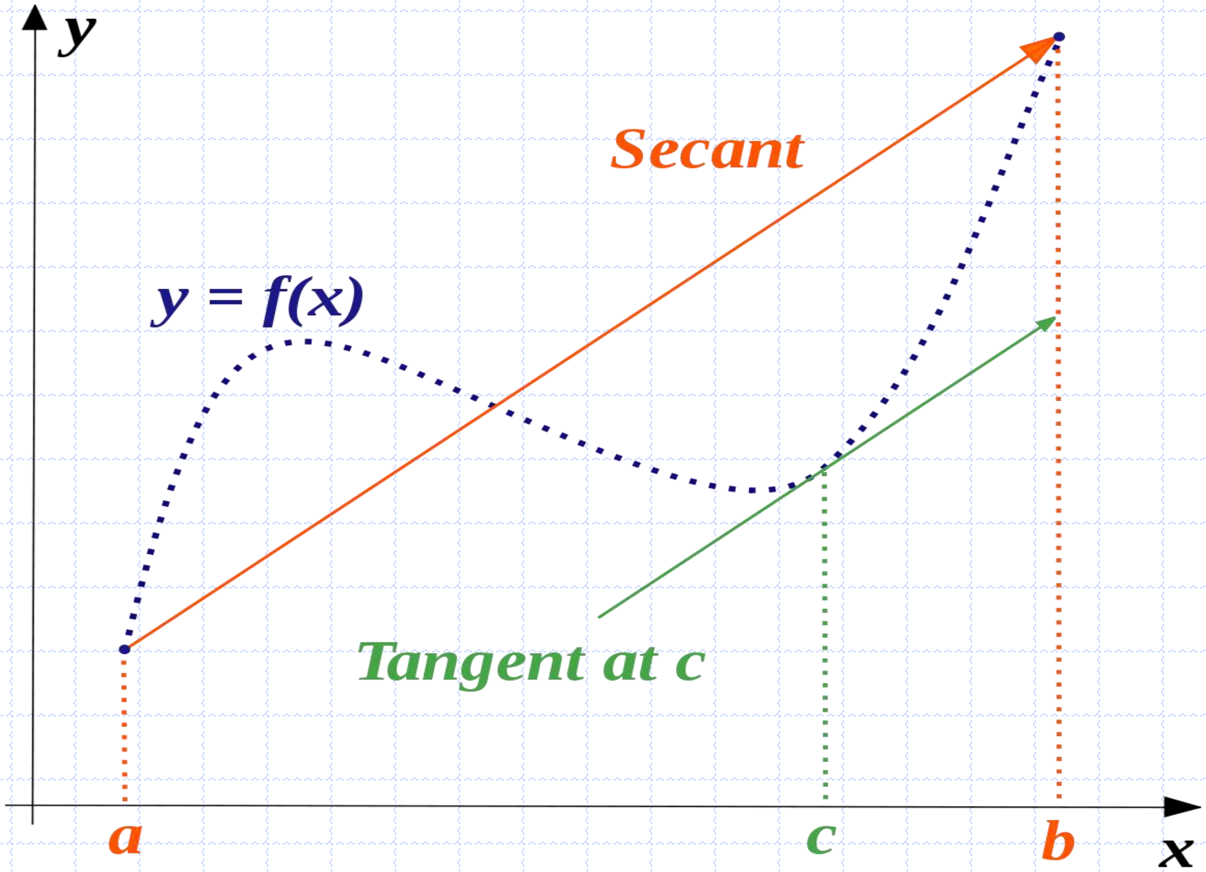
Cauchy's Mean Value Theorem

Let the functions $f(x)$ and $g(x)$ be continuous on an interval $[a, b]$, differentiable on (a, b), and $g'(x)\neq 0$ for all $x\in(a, b)$. Then there is a point $x=c$ in this interval such that

$$[f(b)-f(a)]/[g(b)-g(a)] = f'(c)/g'(c).$$

Geometric meaning

Suppose that a curve γ is described by the parametric equations $x=f(t)$, $y=g(t)$, where the parameter t ranges in the interval $[a,b]$. When changing the parameter t, the point of the curve in Figure given below runs from $A(f(a),g(a))$ to $B(f(b),g(b))$. According to the theorem, there is a point $(f(c),g(c))$ on the curve γ where the tangent is parallel to the chord joining the ends A and B of the curve.



y
Secant
y = f(x)
Tangent at c
a
c
b
x

DEFINITE INTEGRALS

In defining a definite integral $\int_a^b f(x)\,dx$, we dealt with a function *f* defined on a finite interval [*a*, *b*] and we assumed that *f* does not have an infinite discontinuity

Improper Integrals

In this section, we will learn:

How to solve definite integrals

where the interval is infinite and

where the function has an infinite discontinuity.

In this section, we extend the concept of a definite integral to the cases where:

- The interval is infinite
- *f* has an infinite discontinuity in [*a*, *b*]

In either case, the integral is called an improper integral.

Consider the infinite region *S* that lies:

- Under the curve $y = 1/x^2$
- Above the x-axis
- To the right of the line $x = 1$

INFINITE INTERVALS

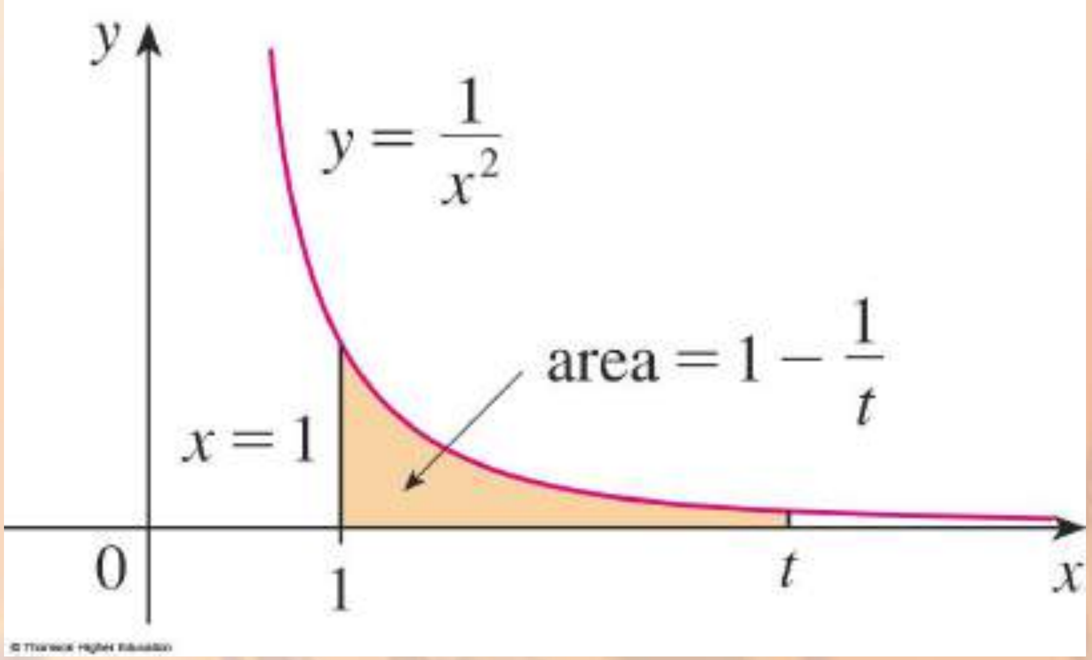
You might think that, since *S* is infinite in extent, its area must be infinite.

- However, let's take a closer look.

The area of the part of *S* that lies to the left of the line *x* = *t* (shaded) is:

$$A(t)=\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1-\frac{1}{t}$$

- Notice that *A*(*t*) < 1 no matter how large *t* is chosen.

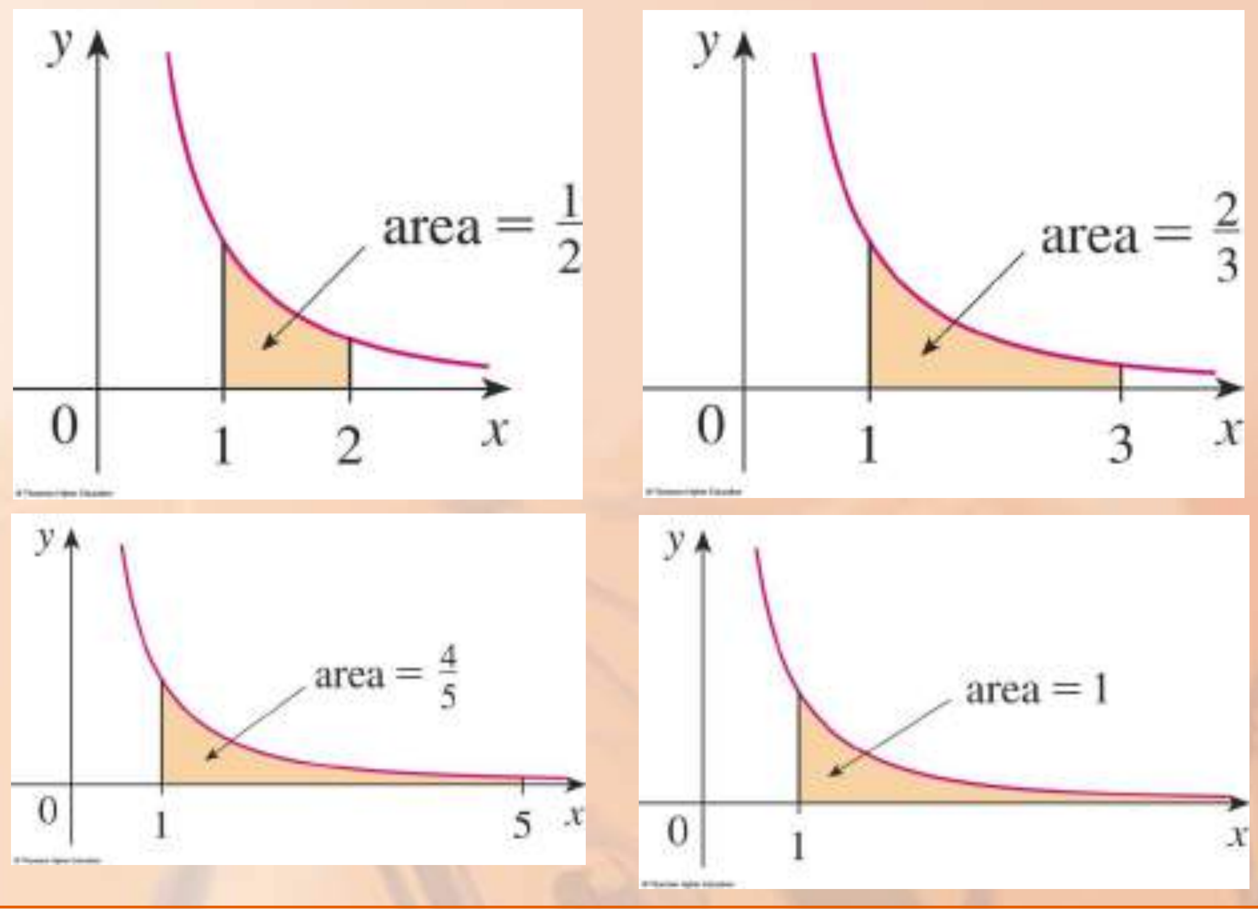


We also observe that:

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(1 - \frac{1}{t}\right) = 1$$

INFINITE INTERVALS

The area of the shaded region approaches 1 as $t \to \infty$.



INFINITE INTERVALS

So, we say that the area of the infinite region *S* is equal to 1 and we write:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$

INFINITE INTERVALS

Using this example as a guide, we define the integral of *f* (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 a**

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 b**

If $\int_t^b f(x)\,dx$ exists for every number $t \leq a$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

CONVERGENT AND DIVERGENT

Definition 1 b

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 c**

If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

- Here, any real number *a* can be used.

IMPROPER INTEGRALS OF TYPE 1

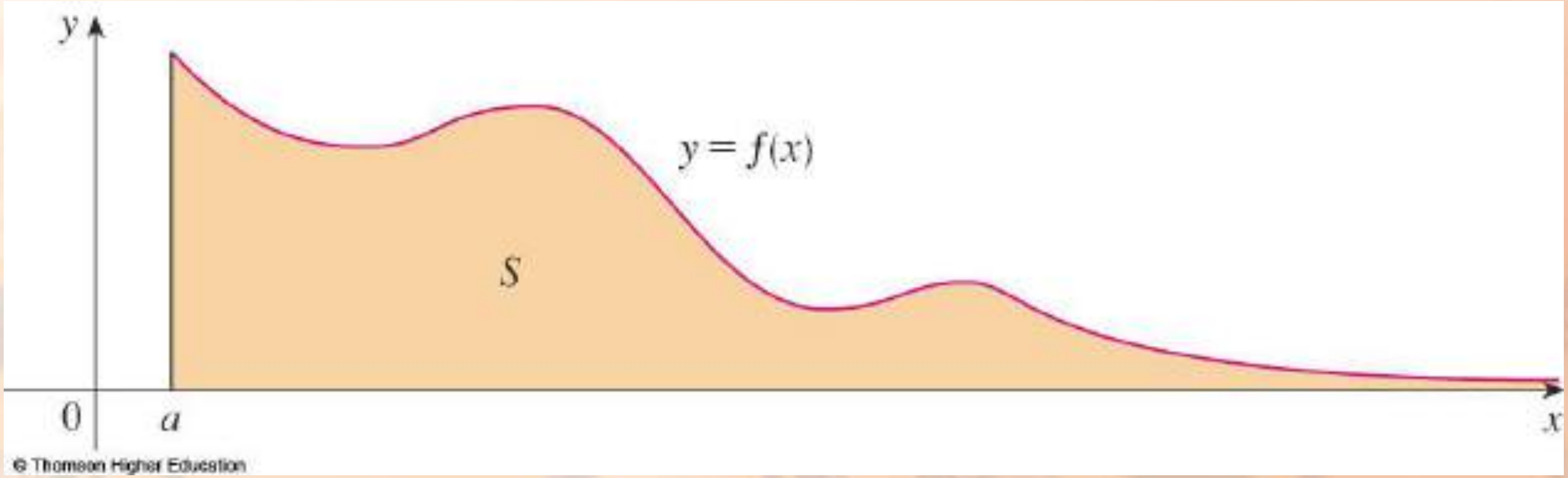
Any of the improper integrals in Definition 1 can be interpreted as an area provided f is a positive function.

IMPROPER INTEGRALS OF TYPE 1

For instance, in case (a), suppose $f(x) \geq 0$ and the integral $\int_a^{\infty} f(x)\,dx$ is convergent.

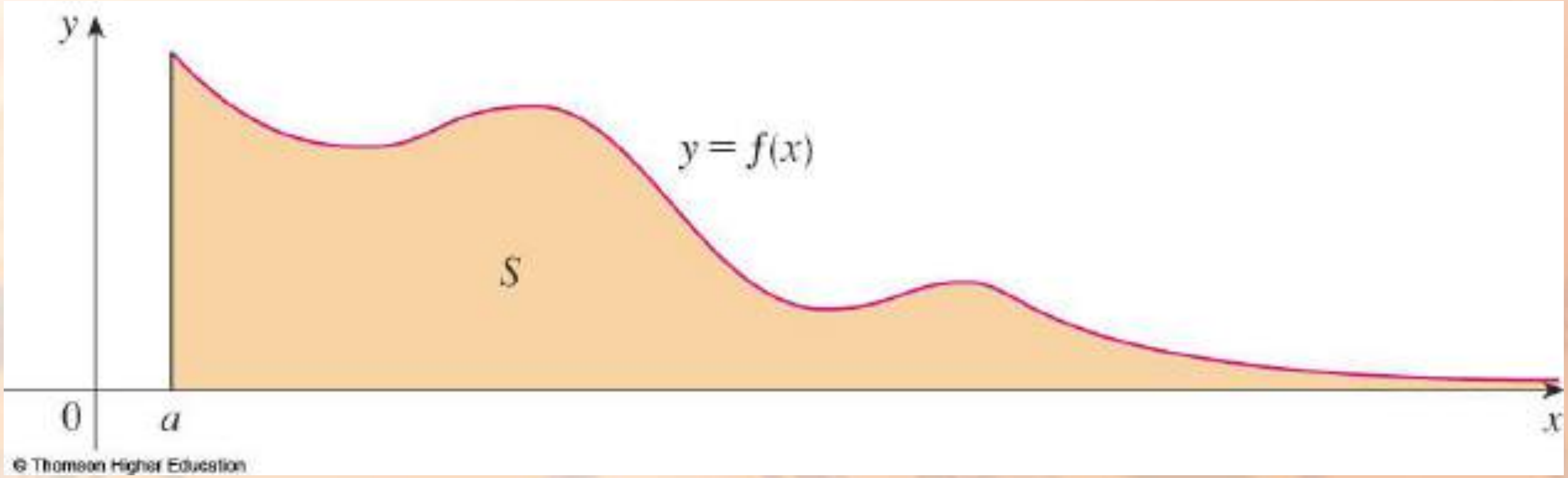
- Then, we define the area of the region $S = \{(x, y) \mid x \geq a,\ 0 \leq y \leq f(x)\}$ in the figure as:

$$A(S) = \int_a^{\infty} f(x)\,dx$$



IMPROPER INTEGRALS OF TYPE 1

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of *f* from *a* to *t*.



Determine whether the integral

$$\int_1^{\infty} (1/x)\, dx$$

is convergent or divergent.

IMPROPER INTEGRALS OF TYPE 1 **Example 1**

According to Definition 1 a,

we have:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}(\ln t - \ln 1)$$

$$= \lim_{t\to\infty}\ln t = \infty$$

- The limit does not exist as a finite number.
- So, the integral is divergent.

IMPROPER INTEGRALS OF TYPE 1

Let's compare the result of Example 1 with the example at the beginning of the section:

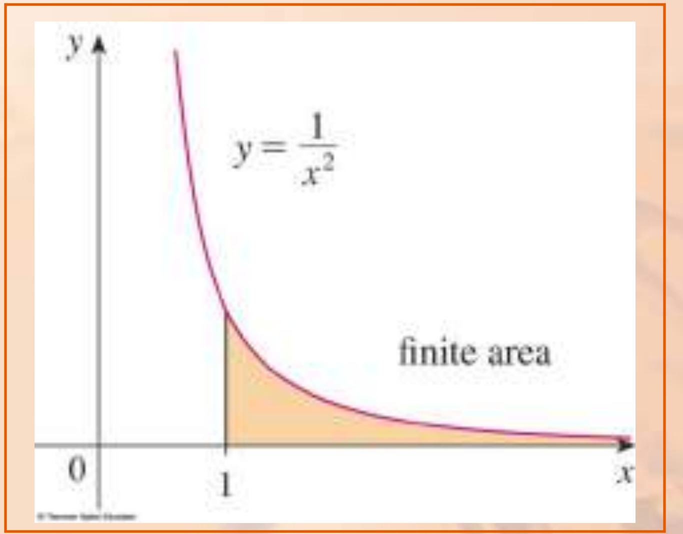
$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

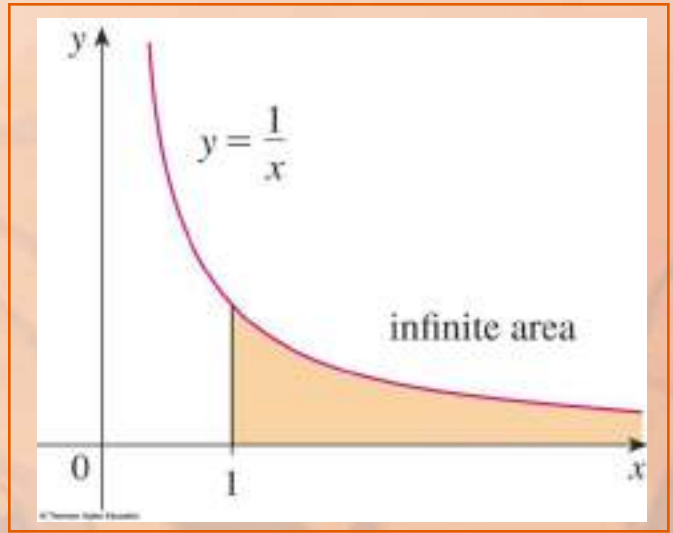
- Geometrically, this means the following.

IMPROPER INTEGRALS OF TYPE 1

The curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$.

However, the region under $y = 1/x^2$ to the right of $x = 1$ has finite area, but the corresponding region under $y = 1/x$ has infinite area.

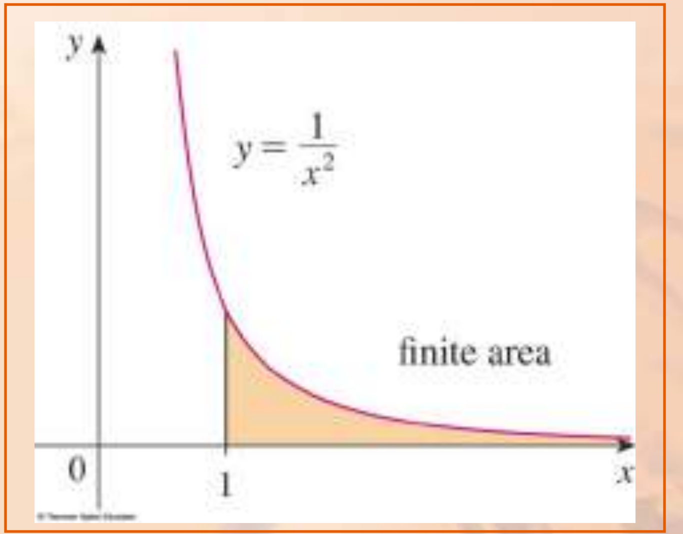


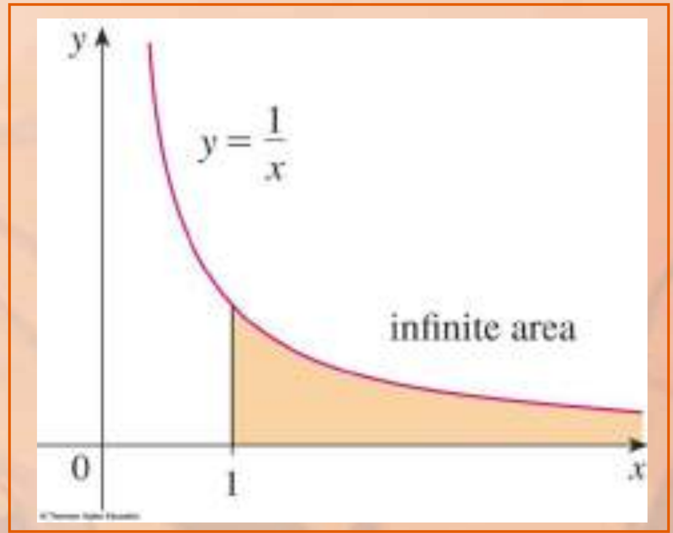


IMPROPER INTEGRALS OF TYPE 1

Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$, but $1/x^2$ approaches faster than $1/x$.

- The values of $1/x$ don't decrease fast enough for its integral to have a finite value.





Example 2

Evaluate $\int_{-\infty}^{0} xe^{x}\,dx$

- Using Definition 1 b, we have: $$\int_{-\infty}^{0} xe^{x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} xe^{x}\,dx$$

 Example 2

- We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = xe^x\Big]_t^0 - \int_t^0 e^x\,dx$$

$$= -te^t - 1 + e^t$$

- We know that $e^t \to 0$ as $t \to -\infty$, and, by l'Hospital's Rule, we have:

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}}$$

$$= \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} (-e^t)$$

$$= 0$$

- Therefore,

$$\int_{-\infty}^{0} xe^{x}\,dx = \lim_{t\to-\infty}\left(-te^{t} - 1 + e^{t}\right)$$

$$= -0 - 1 + 0$$

$$= -1$$

IMPROPER INTEGRALS OF TYPE 1 **Example 3**

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

- It's convenient to choose *a* = 0 in Definition 1 c:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{\infty} \frac{1}{1+x^2}\, dx$$

We must now evaluate the integrals on the right side separately—as follows.

IMPROPER INTEGRALS OF TYPE 1 **Example 3**

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty} \tan^{-1} x \Big]_0^t$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0)$$

$$= \lim_{t\to\infty} \tan^{-1} t$$

$$= \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty} \tan^{-1} x \Big]_t^0$$

$$= \lim_{t\to-\infty} (\tan^{-1} 0 - \tan^{-1} t)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

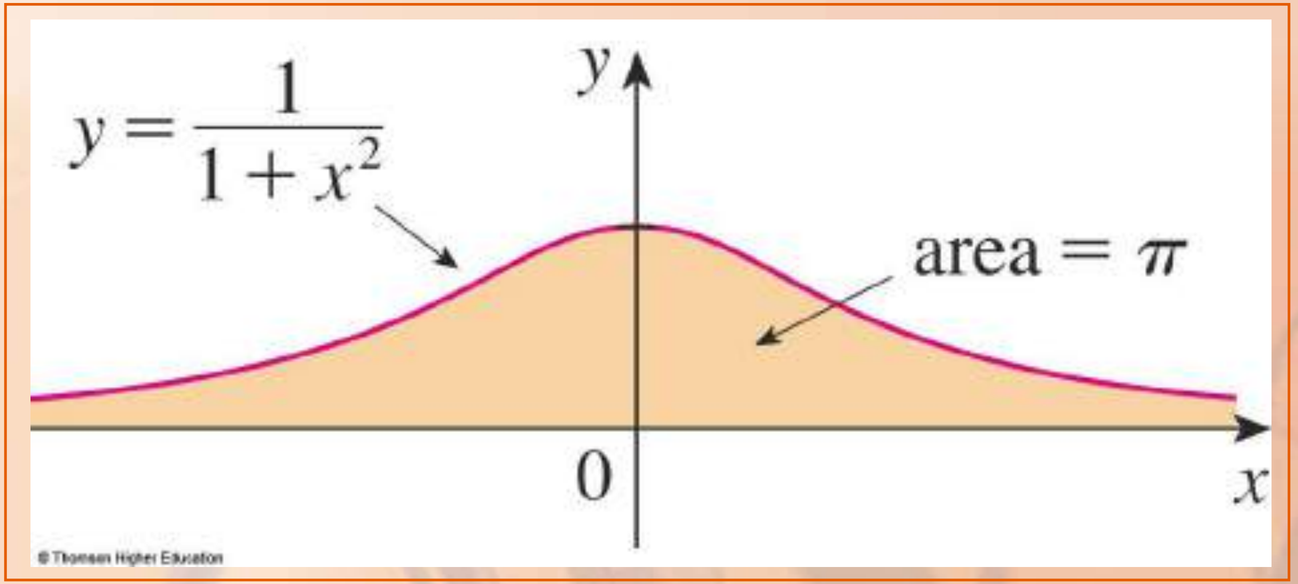
$$= \frac{\pi}{2}$$

Since both these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

IMPROPER INTEGRALS OF TYPE 1 **Example 3**

As $1/(1 + x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x–axis.



 Example 4

For what values of *p* is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$ convergent?

- We know from Example 1 that, if $p = 1$, the integral is divergent.
- So, let's assume that $p \neq 1$.

Then,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $p > 1$, then $p - 1 > 0$.

So, as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.

- Therefore, $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{if } p > 1$
- So, the integral converges.

However, if p <1, then p – 1 < 0.

So, $$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \text{ as } t \to \infty$$

- Thus, the integral diverges.

IMPROPER INTEGRALS OF TYPE 1 **Definition 2**

We summarize the result of Example 4 for future reference:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is:}$$

- Convergent if $p > 1$
- Divergent if $p \le 1$

TYPE 2—DISCONTINUOUS INTEGRANDS

Suppose *f* is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at *b*.

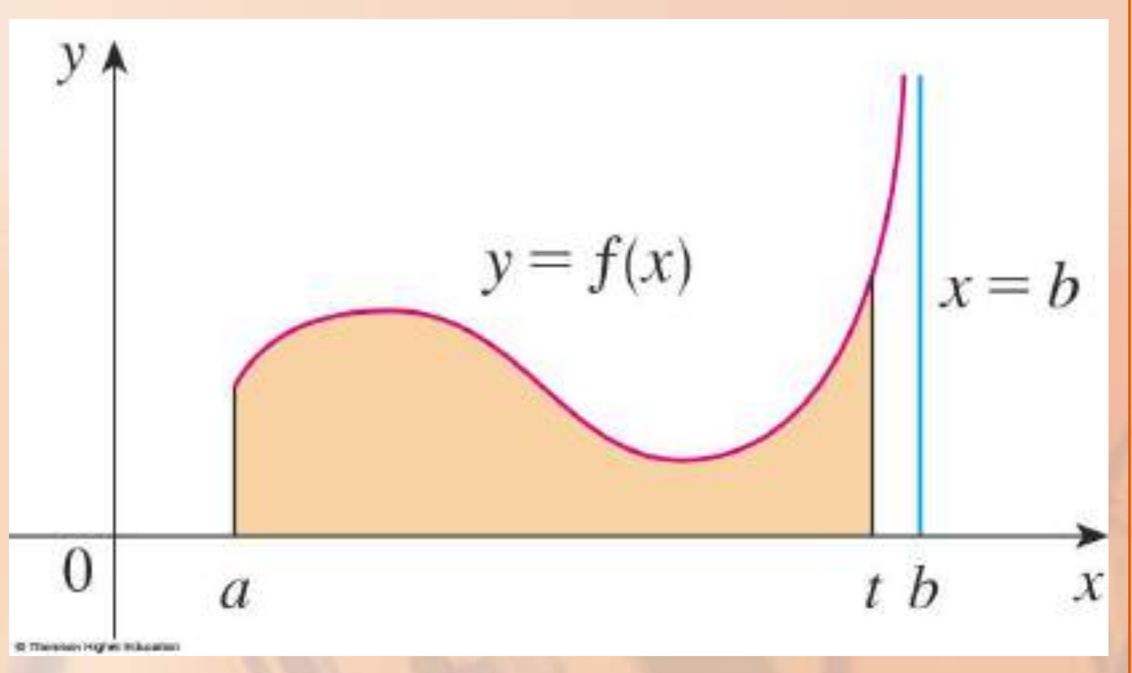
DISCONTINUOUS INTEGRANDS

Let S be the unbounded region under the graph of f and above the x-axis between a and b.

- For Type 1 integrals, the regions extended indefinitely in a horizontal direction.
- Here, the region is infinite in a vertical direction.

The area of the part of *S* between *a* and *t* (shaded region) is:

$$A(t) = \int_a^t f(x)\,dx$$



DISCONTINUOUS INTEGRANDS

If it happens that $A(t)$ approaches a definite number A as $t \rightarrow b^-$, then we say that the area of the region S is A and we write:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

DISCONTINUOUS INTEGRANDS

We use the equation to define an improper integral of Type 2 even when *f* is not a positive function—no matter what type of discontinuity *f* has at *b*.

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 a**

If *f* is continuous on [*a*, *b*) and is discontinuous at *b*, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

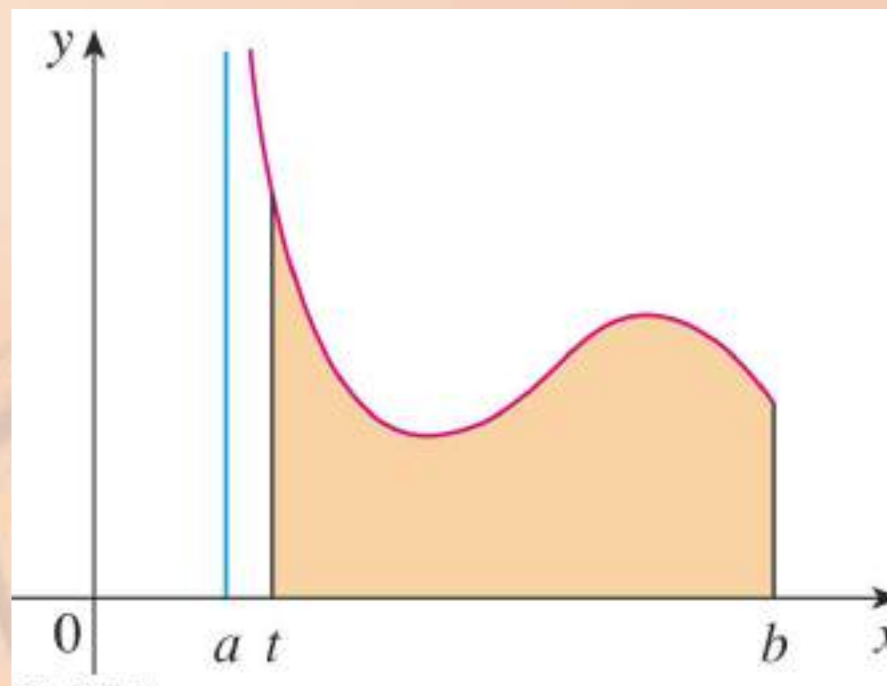
If *f* is continuous on (*a*, *b*] and is discontinuous at *a*, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

Definition 3 b is illustrated for the case where $f(x) \geq 0$ and has vertical asymptotes at a and c, respectively.



IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

The improper integral $\int_a^b f(x)\,dx$

is called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

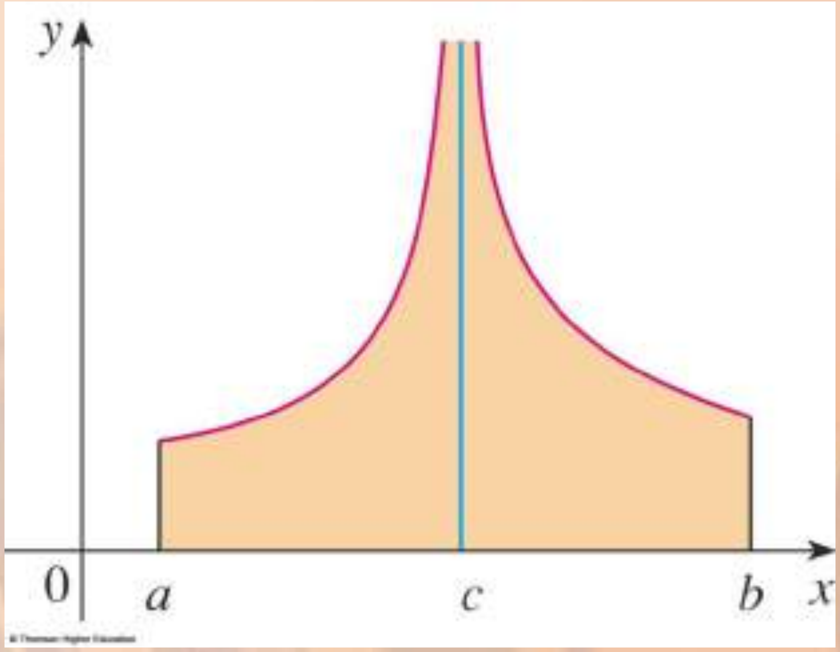
IMPROPER INTEGRAL OF TYPE 2 **Definition 3 c**

If *f* has a discontinuity at *c*, where *a* < *c* < *b*, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 c**

Definition 3 c is illustrated for the case where $f(x) \geq 0$ and has vertical asymptotes at a and c, respectively.



IMPROPER INTEGRALS OF TYPE 2 **Example 5**

Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

- First, we note that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$.

IMPROPER INTEGRALS OF TYPE 2 **Example 5**

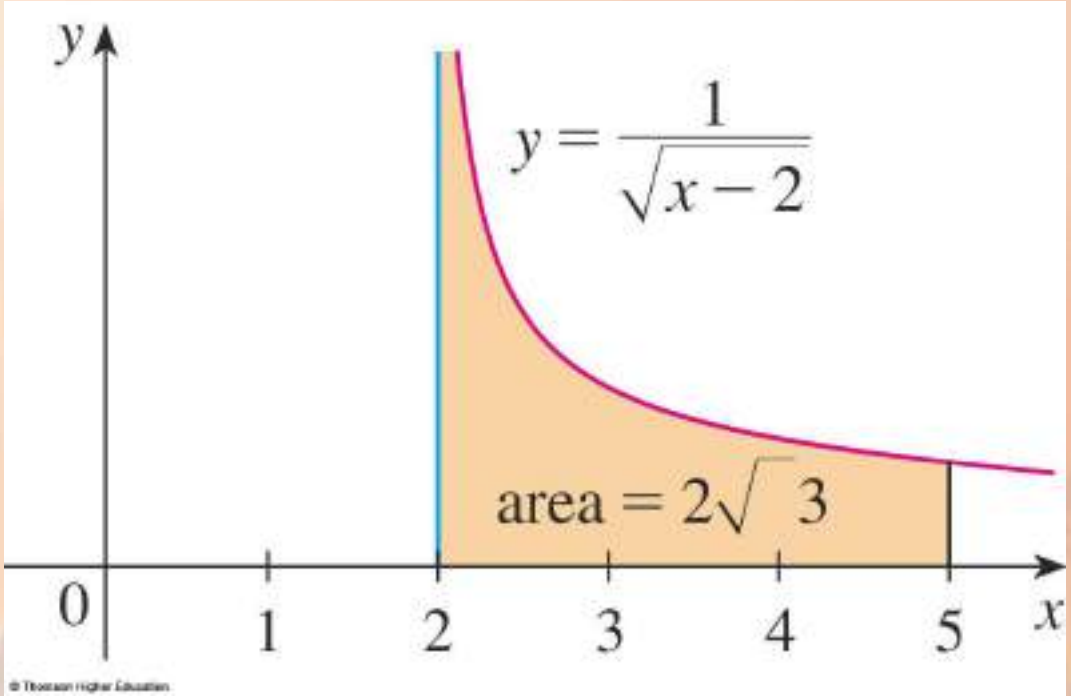
- The infinite discontinuity occurs at the left end-point of [2, 5].
- So, we use Definition 3 b:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5 \\ &= \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2}) \\ &= 2\sqrt{3}\end{aligned}$$

- Thus, the given improper integral is convergent.

IMPROPER INTEGRALS OF TYPE 2 **Example 5**

- Since the integrand is positive, we can interpret the value of the integral as the area of the shaded region here.



Determine whether $\int_0^{\pi/2} \sec x \, dx$ converges or diverges.

- Note that the given integral is improper because: $\lim_{x \to (\pi/2)^-} \sec x = \infty$

IMPROPER INTEGRALS OF TYPE 2 **Example 6**

- Using Definition 2 a, we have:

$$\int_0^{\pi/2} \sec x\, dx = \lim_{x \to (\pi/2)^-} \int_0^t \sec x\, dx$$

$$= \lim_{x \to (\pi/2)^-} \ln\left|\sec x + \tan x\right| \Big]_0^t$$

$$= \lim_{x \to (\pi/2)^-} \left[\ln(\sec t + \tan t) - \ln 1\right] = \infty$$

- This is because sec $t \to \infty$ and tan $t \to \infty$ as $t \to (\pi/2)^-$.

- Thus, the given improper integral is divergent.

IMPROPER INTEGRALS OF TYPE 2 **Example 7**

Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

- Observe that the line $x = 1$ is a vertical asymptote of the integrand.

IMPROPER INTEGRALS OF TYPE 2 **Example 7**

- As it occurs in the middle of the interval [0, 3], we must use Definition 3 c with *c* = 1:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} |x-1| \Big]_0^t$$

$$= \lim_{t\to 1^-} (\ln|t-1| - \ln|-1|)$$

$$= \lim_{t\to 1^-} \ln(1-t) = -\infty$$

- This is because $1 - t \to 0^+$ as $t \to 1^-$.

IMPROPER INTEGRALS OF TYPE 2 **Example 7**

Thus, $\int_0^1 dx/(x-1)$ is divergent.

This implies that $\int_0^3 dx/(x-1)$ is divergent.

- We do not need to evaluate $\int_1^3 dx/(x-1)$.

WARNING

Then, we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

- This is wrong because the integral is improper and must be calculated in terms of limits.

WARNING

From now, whenever you meet the symbol $\int_a^b f(x)\,dx$, you must decide, by looking at the function *f* on [*a*, *b*], whether it is either:

- An ordinary definite integral
- An improper integral

 Example 8

Evaluate $\int_0^1 \ln x \, dx$

- We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x \to 0^+} \ln x = -\infty$.
- Thus, the given integral is improper, and we have:

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

IMPROPER INTEGRALS OF TYPE 2 **Example 8**

- Now, we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x \, dx = x \ln x \Big]_t^1 - \int_t^1 dx$$

$$= 1 \ln 1 - t \ln t - (1 - t)$$

$$= -t \ln t - 1 + t$$

IMPROPER INTEGRALS OF TYPE 2 **Example 8**

- To find the limit of the first term, we use l'Hospital's Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t}$$

$$= \lim_{t\to 0^+} \frac{1/t}{-1/t^2}$$

$$= \lim_{t\to 0^+}(-t)$$

$$= 0$$

- Therefore,

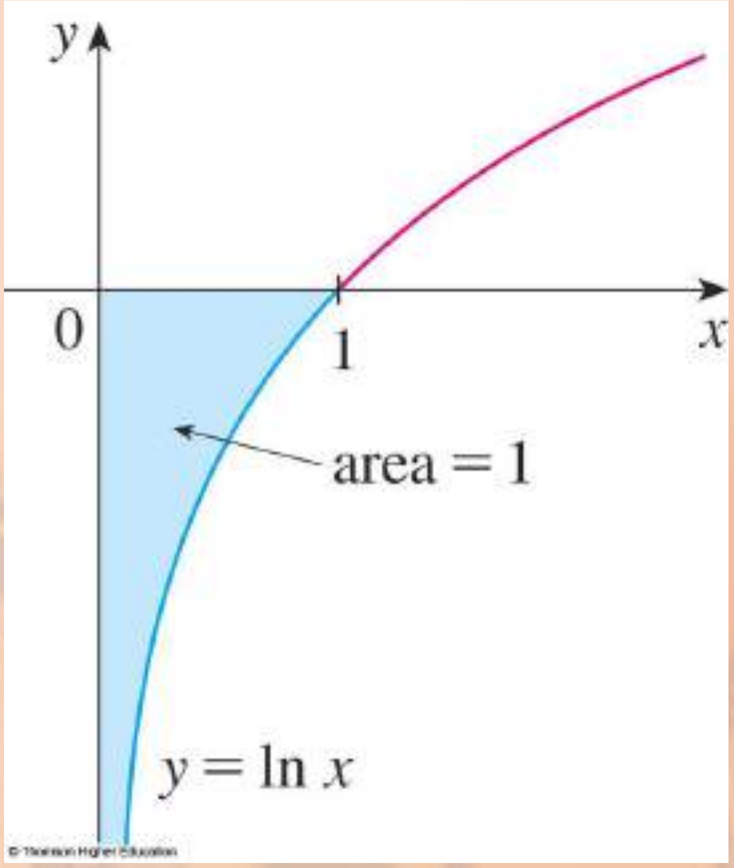
$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}(-t\ln t - 1 + t)$$

$$= -0 - 1 + 0$$

$$= -1$$

The geometric interpretation of the result is shown.

- The area of the shaded region above $y = \ln x$ and below the x-axis is 1.



A COMPARISON TEST FOR IMPROPER INTEGRALS

Sometimes, it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent.

- In such cases, the following theorem is useful.
- Although we state it for Type 1 integrals,
  a similar theorem is true for Type 2 integrals.

COMPARISON THEOREM

Suppose *f* and *g* are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

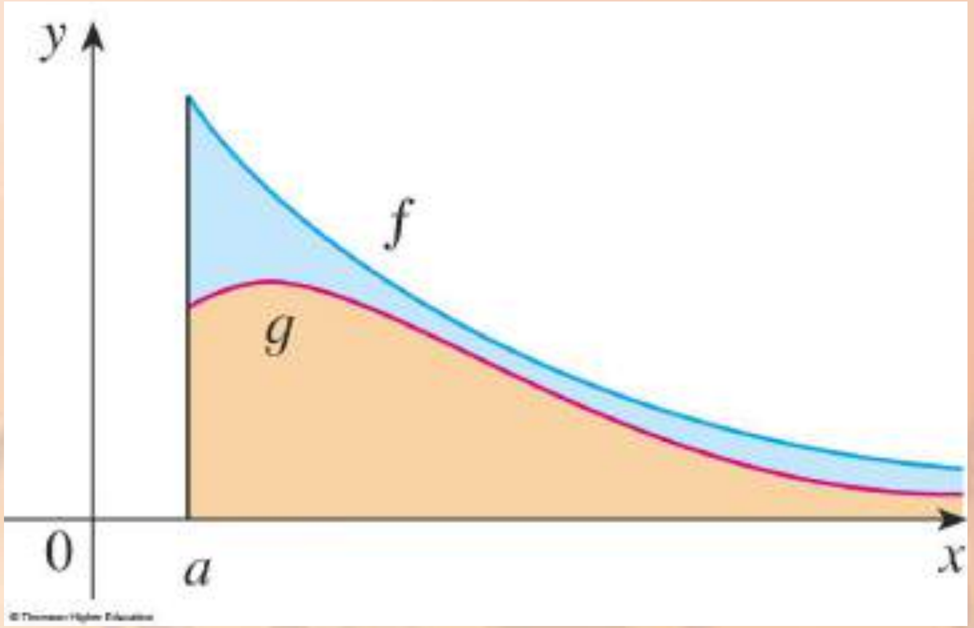
a. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

b. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

COMPARISON THEOREM

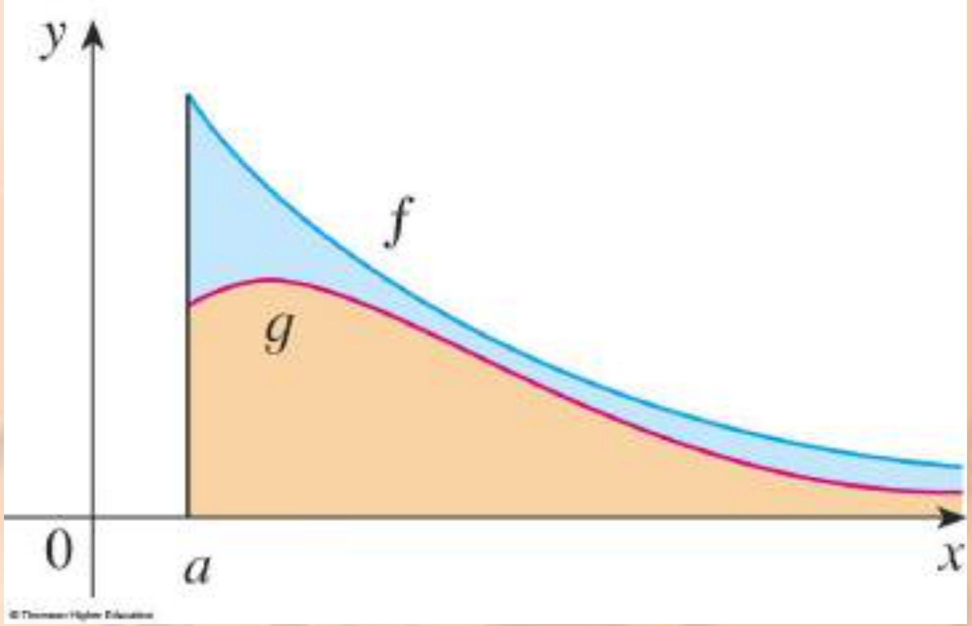
We omit the proof of the theorem.

However, the figure makes it seem plausible.



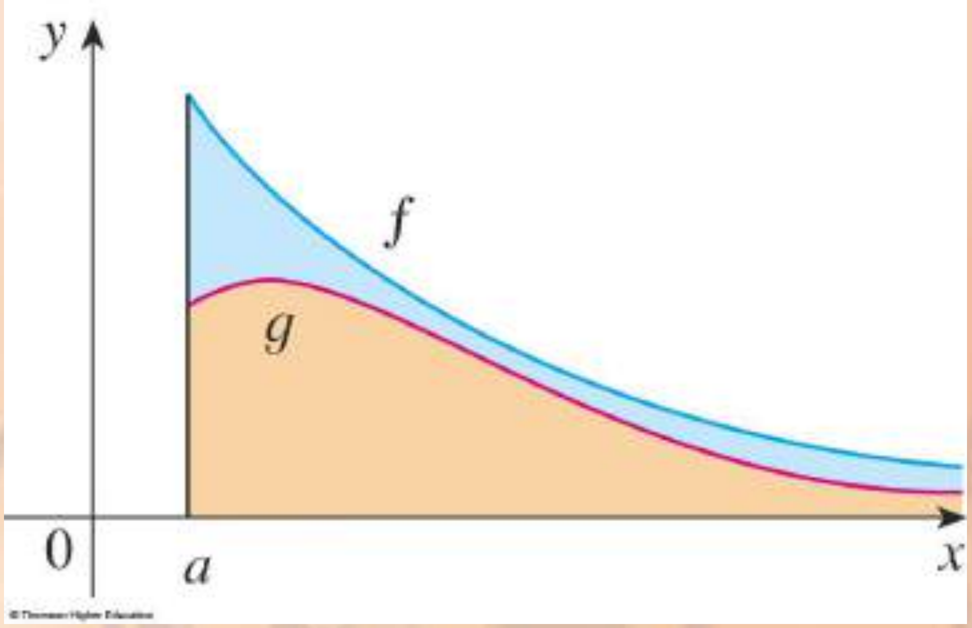
COMPARISON THEOREM

If the area under the top curve $y = f(x)$ is finite, so is the area under the bottom curve $y = g(x)$.



COMPARISON THEOREM

If the area under $y = g(x)$ is infinite, so is the area under $y = f(x)$.



COMPARISON THEOREM

Note that the reverse is not necessarily true:

- If $\int_a^\infty g(x)\,dx$ is convergent, $\int_a^\infty f(x)\,dx$ may or may not be convergent.
- If $\int_a^\infty f(x)\,dx$ is divergent, $\int_a^\infty g(x)\,dx$ may or may not be divergent.

COMPARISON THEOREM Example 9

Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

- We can't evaluate the integral directly.
- The antiderivative of e^{-x^2} is not an elementary function (as explained in Section 7.5).

COMPARISON THEOREM — Example 9

We write:

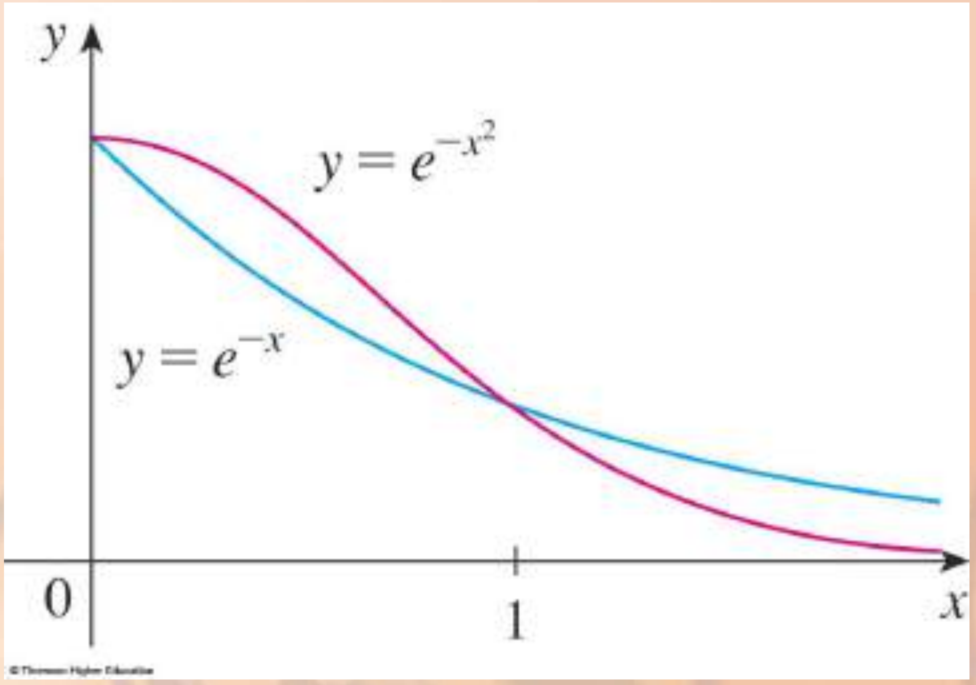
$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

- We observe that the first integral on the right-hand side is just an ordinary definite integral.

COMPARISON THEOREM Example 9

- In the second integral, we use the fact that, for $x \geq 1$, we have $x^2 \geq x$.

- So, $-x^2 \leq -x$ and, therefore, $e^{-x^2} \leq e^{-x}$.



The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty}(e^{-1} - e^{-t})$$

$$= e^{-1}$$

Thus, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the theorem, we see that $\int_1^{\infty} e^{-x^2}\,dx$ is convergent.

- It follows that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

COMPARISON THEOREM

In Example 9, we showed that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent without computing its value.

- In Exercise 70, we indicate how to show that its value is approximately 0.8862
- In probability theory, it is important to know the exact value of this improper integral.
- Using the methods of multivariable calculus, it can be shown that the exact value is $\sqrt{\pi}/2$.

COMPARISON THEOREM

The table illustrates the definition of an improper integral by showing how the (computer- generated) values of $\int_0^t e^{-x^2}\,dx$ approach $\sqrt{\pi}/2$ as *t* becomes large.

- In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

TABLE I

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

The integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem since $\frac{1+e^{-x}}{x} > \frac{1}{x}$

$\int_1^\infty (1/x)\,dx$ is divergent by Example 1 or by Definition 2 with p = 1.

COMPARISON THEOREM

The table illustrates the divergence of the integral in Example 10.

- It appears the values are not approaching any fixed number.

TABLE 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255