Ratio Test

Ratio Test

If $\sum a_n$ is a positive-term series and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ then

i. $\sum a_n$ is convergent if $L<1$
ii. $\sum a_n$ is divergent if $L>1$
iii. The test is inconclusive if $L=1$

Example 1

Use the ratio test to determine whether the following series converge or diverge.

(a) $\sum_{k=1}^{\infty} \frac{1}{k!}$

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \frac{1}{(k+1)!} \cdot \frac{k!}{1} = \frac{k!}{(k+1)k!} = \frac{1}{k+1} = 0 < 1$$

$converges$

Example 2

Use the ratio test to determine whether the following series converge or diverge.

(b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{2^k(k+1)}{2 \cdot 2^k \cdot k} = \frac{k+1}{2k} = \frac{1}{2} < 1$$

converges

Example 3

Use the ratio test to determine whether the following series converge or diverge.

(c) $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{k!}$

$$\rho=\lim_{k\to+\infty}\frac{u_{k+1}}{u_k}=\frac{(k+1)^{k+1}}{(k+1)!}\cdot\frac{k!}{k^k}=\frac{(k+1)^k}{k^k}=\lim_{k\to+\infty}\left(1+\tfrac{1}{k}\right)^k=e>1$$

$diverges$

Example 4

Use the ratio test to determine whether the following series converge or diverge.

(d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \frac{(2k+2)(2k+1)}{4} = +\infty > 1$$

diverges

Root Test

Let $\sum U_n$ be a non negative series, and assume that $\lim_{n\to\infty} \sqrt[n]{U_n} = L$ (possibly ∞)

a. If $0 \le L < 1$, then $\sum U_n$ converges.

b. If $L > 1$, then $\sum U_n$ diverges.

If $L=1$ then from this test alone we can not draw any conclusion about the convergence or divergence of $\sum U_n$.

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution

Let
$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\log n}$$

$$= 0 < 1. \qquad [\because \log\infty \to \infty]$$

Hence, by Cauchy's root test, the series is convergent.

Example 2: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{2}$$

$$\lim_{n\to\infty} \sqrt[n]{n^2} = \left(\lim_{n\to\infty} \sqrt[n]{n}\right)^2$$

$$= 1^2 \quad = 1$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{2}$$

$$= \frac{1}{2}$$

$\therefore$ it converges

Thank you

Indeterminate Forms

Indeterminate Forms

- What are indeterminate forms?
  - In calculus and other branches of mathematical analysis, limits involving an algebraic combination of functions in an independent variable may often be evaluated by replacing these functions by their limits.
  - If the expression obtained after this substitution does not give enough information to determine the original limit, it is said to take on an *indeterminate form.*

Types of Indeterminate forms

- There are seven types of indeterminate forms :

1. $0/0$
2. ∞/∞
3. $0 \times \infty$
4. $\infty - \infty$
5. 0^0
6. 1^∞
7. ∞^0

0/0 Form

- Limit of the form $\frac{f(x)}{g(x)}$, where $\lim f(x) = \lim g(x) = 0$ are called indeterminate form of the type 0/0.

Consider: $\lim_{x\to 2}\frac{x^2-4}{x-2}$

If we try to evaluate this by direct substitution, we get: $\frac{0}{0}$

Zero divided by zero can not be evaluated, and is an example of **indeterminate form**.

In this case, we can evaluate this limit by factoring and canceling:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4$$

L' Hopital's Rule

- L'Hopital's rule is a general method for evaluating the indeterminate forms 0/0 and ∞/∞. This rule states that (under appropriate conditions)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

  where f' and g' are the derivatives of f and g.
- Note that this rule does *not* apply to expressions ∞/0, 1/0, and so on.
- These derivatives will allow one to perform algebraic simplification and eventually evaluate the limit.

L' Hopital's Rule

- Rules to evaluate 0/0 form :
  1. Check whether the limit is an indeterminate form. If it is not, then we cannot apply L' Hopital's rule.
  2. Differentiate $f(x)$ and $g(x)$ separately.
  3. If $g'(a) \neq 0$, then the limit will exist. It may be finite, $+\infty$ or $-\infty$. If $g'(a)=0$ then follow rule 4.
  4. Differentiate $f'(x)$ & $g'(x)$ separately.
  5. Continue the process till required value is reached.

Example:

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} = \lim_{x\to 0}\frac{\sin x}{1+2x} = 0$$

If it's no longer indeterminate, then **STOP**!

If we try to continue with L'Hôpital's rule:

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

which is wrong, wrong, wrong!

On the other hand, you can apply L'Hôpital's rule as many times as necessary as long as the fraction is still indeterminate:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \longleftarrow \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \longleftarrow \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \longleftarrow \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}$$

$$=-\frac{1}{8}$$

Example 1: $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{2x}{1} = 2(2) = 4$

Example 2: $\lim_{x \to 0} \frac{\tan 3x}{\sin 2x} = \lim_{x \to 0} \frac{3\sec^2 3x}{2\cos 2x} = \frac{3(1)}{2(1)} = \frac{3}{2}$

Example 3: $\lim_{h \to 0} \frac{\sqrt[3]{8+h}-2}{h} = \lim_{h \to 0} \frac{\frac{1}{3}(8+h)^{-2/3}(1)}{1} = \lim_{h \to 0} \frac{1}{3(8+h)^{2/3}} = \frac{1}{3(8)^{2/3}} = \frac{1}{12}$

Example 4: $\lim_{x \to \pi/3} \frac{\cos x - 1/2}{x - \pi/3} = \lim_{x \to \pi/3} \frac{-\sin x}{1} = -\sin\left(\pi/3\right) = -\frac{\sqrt{3}}{2}$

∞ / ∞ Form

- If $\lim_{x \to c} f(x) = \infty, \lim_{x \to c} g(x) = \infty$, then it is indeterminate form of type 0/0.

EXAMPLES:

1. Find $\lim_{x \to \infty} \frac{5x - 2}{7x + 3}$.

Solution 1: We have

$$\lim_{x \to \infty} \frac{5x - 2}{7x + 3} = \left[\frac{\infty}{\infty}\right] = \lim_{x \to \infty} \frac{\frac{5x-2}{x}}{\frac{7x+3}{x}} = \lim_{x \to \infty} \frac{\frac{5x}{x} - \frac{2}{x}}{\frac{7x}{x} + \frac{3}{x}} = \lim_{x \to \infty} \frac{5 - \frac{2}{x}}{7 + \frac{3}{x}} = \frac{5 - 0}{7 + 0} = \frac{5}{7}$$

0x∞ Form

- Limit of the form $\lim_{x \to c} f(x) = 0,\ \lim_{x \to c} g(x) = \infty$ are called indeterminate form of the type 0x∞.
- If we write f(x) g(x) = f(x)/[1/g(x)], then the limit becomes of the form (0/0).
- This can be evaluated by using L' Hopital's rule.

Example

Example 3.1. Consider $\lim_{x\to\infty}\left(x\cdot\sin\left(\frac{2}{x}\right)\right)$. This has the form $\infty\cdot 0$ if you try to evaluate directly. But if you flip the x to the denominator

$$\lim_{x\to\infty}\left(\frac{\sin\left(\frac{2}{x}\right)}{1/x}\right)$$

then this limit has the form $\frac{0}{0}$. We have simply taken the ∞, and transformed it into a 0 in the denominator. This limit can be done with L'Hôpital's rule.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{\sin\left(\frac{2}{x}\right)}{1/x}\right) &= \lim_{x\to\infty}\frac{\cos\left(\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)}{-1/x^2} \\
&= \lim_{x\to\infty} 2\cos\left(\frac{2}{x}\right) \\
&= 2\cos(2/\infty) \\
&= 2\cos(0) \\
&= 2
\end{aligned}$$

So in this case, we could evaluate the limit by flipping it to $\frac{0}{0}$ and using L'Hôpital's rule.

$\infty - \infty$ Form

- Limit of the form $\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$ are called indeterminate form of the type ∞-∞.
- If we write $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} \frac{1/g(x) - 1/f(x)}{1/(f(x)g(x))}$, then the limit becomes of the form (0/0) and can be evaluated by using the L' Hopital's rule.

Example

11. Find $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

Solution: We have

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=[\infty-\infty]=\lim_{x\to 1}\left(\frac{1\cdot(x-1)}{\ln x\cdot(x-1)}-\frac{\ln x\cdot 1}{\ln x\cdot(x-1)}\right)$$

$$=\lim_{x\to 1}\frac{x-1-\ln x}{\ln x(x-1)}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1-\ln x)'}{(\ln x(x-1))'}$$

$$=\lim_{x\to 1}\frac{x'-1'-(\ln x)'}{(\ln x)'\cdot(x-1)+\ln x\cdot(x-1)'}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}$$

$$=\lim_{x\to 1}\frac{\left(1-\frac{1}{x}\right)x}{\left(\frac{x-1}{x}+\ln x\right)x}=\lim_{x\to 1}\frac{1\cdot x-\frac{1}{x}\cdot x}{\frac{x-1}{x}\cdot x+\ln x\cdot x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1)'}{(x-1+x\ln x)'}=\lim_{x\to 1}\frac{x'-1'}{x'-1'+x'\ln x+x(\ln x)'}$$

$$=\lim_{x\to 1}\frac{1-0}{1-0+1\cdot\ln x+x\cdot\frac{1}{x}}=\lim_{x\to 1}\frac{1}{2+\ln x}=\frac{1}{2+0}=\frac{1}{2}$$

0^0 Form

- Limit of the form $\lim_{x\to c} f(x) = 0^+, \lim_{x\to c} g(x) = 0$ are called indeterminate form of the type 0^0.

- If we write $\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$, then the limit becomes of the form (0/0) and can be evaluated by using the L’ Hopital’s rule.

Example

Example 1: Find $\lim_{x\to 0^+} x^x$.

This is an indeterminate form of the type 0^0. Let $y = x^x \Rightarrow \ln y = \ln x^x =$

$x\ln x$. $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$

Thus, $\lim_{x\to 0^+} x^x = e^0 = 1$.

1^∞ Form

- Limit of the form $\lim_{x\to c} f(x) = 1,\ \lim_{x\to c} g(x) = \infty$ are called indeterminate form of the type 1^∞ .
- If we write $\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$, then the limit becomes of the form (0/0) and can be evaluated by using the L′ Hopital’s rule.

Example

Example 3: Find $\lim\limits_{x \to 0^+} (\cos x)^{1/x}$.

This is an indeterminate form of the type 1^∞. Let $y = (\cos x)^{1/x} \Rightarrow$ $\ln y = \ln\left[(\cos x)^{1/x}\right] = \dfrac{\ln(\cos x)}{x}$. $\lim\limits_{x \to 0^+} \ln y = \lim\limits_{x \to 0^+} \dfrac{\ln(\cos x)}{x} =$ $\lim\limits_{x \to 0^+}(-\tan x) = 0$. Thus, $\lim\limits_{x \to 0^+} (\cos x)^{1/x} = e^0 = 1$.

∞^0 Form

- Limit of the form $\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = 0$ are called indeterminate form of the type ∞^0 .
- If we write $\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$, then the limit becomes of the form (0/0) and can be evaluated by using the L′ Hopital′s rule.

Example

Example 2: Find $\lim_{x \to +\infty} (e^x + 1)^{-2/x}$.

This is an indeterminate form of the type ∞^0. Let $y = (e^x + 1)^{-2/x} \Rightarrow$
$\ln y = \ln\left[(e^x + 1)^{-2/x}\right] = \dfrac{-2\ln(e^x + 1)}{x}$. $\lim_{x \to +\infty} \ln y = \lim_{x \to +\infty} \dfrac{-2\ln(e^x + 1)}{x} =$
$\lim_{x \to +\infty} \dfrac{-2\left(\dfrac{e^x}{e^x + 1}\right)}{1} = \lim_{x \to +\infty} \dfrac{-2e^x}{e^x + 1} = \lim_{x \to +\infty} \dfrac{-2e^x}{e^x} = -2$. Thus, $\lim_{x \to +\infty} (e^x + 1)^{-2/x} =$

e^{-2}.

The following table lists the most common indeterminate forms and the transformations for applying l'Hôpital's rule.

Indeterminate form	Conditions	Transformation to 0/0	Transformation to ∞/∞
0/0	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=0$	–	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$
∞/∞	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$	–
0 × ∞	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{g(x)}{1/f(x)}$
∞ – ∞	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}(f(x)-g(x))=\lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x)-g(x))=\ln\lim_{x\to c}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x)=0^+, \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x)=1,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$

Thank You

5.1 EIGENVECTORS AND EIGENVALUES

EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathrm{x} = \lambda\mathrm{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathrm{x} = \lambda\mathrm{x}$; such an **x** is called an *eigenvector corresponding to λ*.
- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathrm{x} = 0 \quad \text{----(1)}$$

  has a nontrivial solution.
- The set of *all* solutions of (1) is just the null space of the matrix $A - \lambda I$.

EIGENVECTORS AND EIGENVALUES

- So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ.
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.
- **Example 1:** Show that 7 is an eigenvalue of matrix

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

EIGENVECTORS AND EIGENVALUES

- **Solution:** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \qquad \text{----(2)}$$

has a nontrivial solution.

- But (2) is equivalent to $A\mathbf{x} - 7\mathbf{x} = 0$, or

$$(A - 7I)\mathbf{x} = 0 \qquad \text{----(3)}$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

EIGENVECTORS AND EIGENVALUES

- The columns of $A-7I$ are obviously linearly dependent, so (3) has nontrivial solutions.
- To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.

EIGENVECTORS AND EIGENVALUES

- **Example 2:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
- **Solution:** Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathrm{x} = 0$.

EIGENVECTORS AND EIGENVALUES

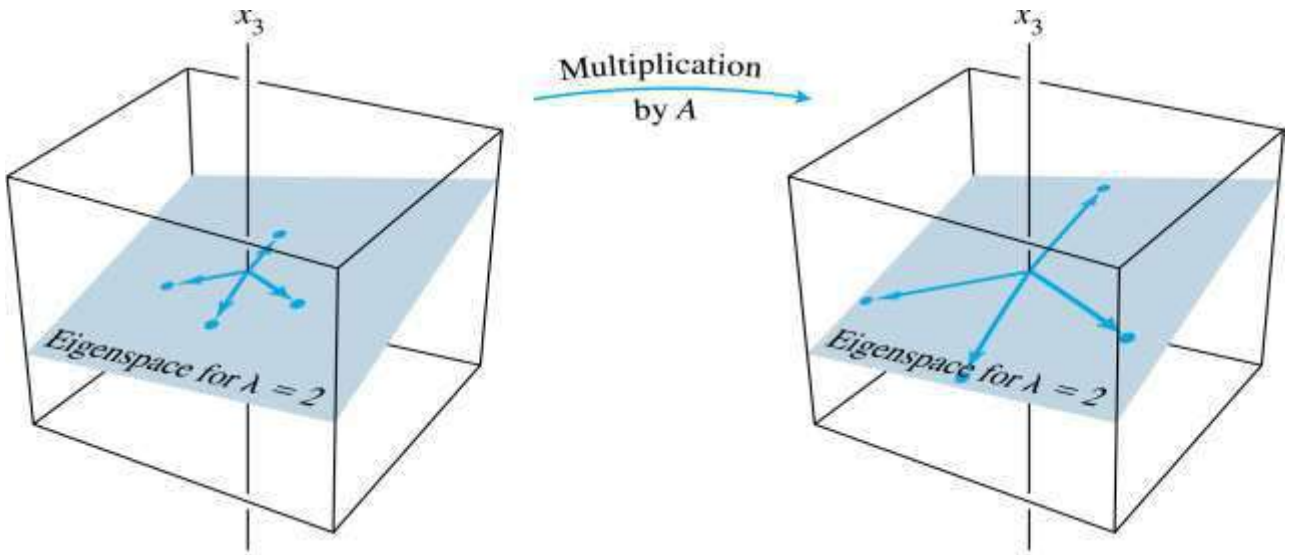
$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathrm{x}=0$ has free variables.
- The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \ x_2 \text{ and } x_3 \text{ free.}$$

EIGENVECTORS AND EIGENVALUES

- The eigenspace, shown in the following figure, is a two-dimensional subspace of $\mathbb{R}^3$.



A acts as a dilation on the eigenspace.

- A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

EIGENVECTORS AND EIGENVALUES

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Proof:** For simplicity, consider the 3×3 case.
- If A is upper triangular, the $A - \lambda I$ has the form

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

EIGENVECTORS AND EIGENVALUES

- The scalar λ is an eigenvalue of A if and only if the equation $(A-\lambda I)\mathrm{x}=0$ has a nontrivial solution, that is, if and only if the equation has a free variable.

- Because of the zero entries in $A-\lambda I$, it is easy to see that $(A-\lambda I)\mathrm{x}=0$ has a free variable if and only if at least one of the entries on the diagonal of $A-\lambda I$ is zero.

- This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A.

EIGENVECTORS AND EIGENVALUES

- **Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.
- **Proof:** Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent.
- Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors.
- Let p be the least index such that v_{p+1} is a linear combination of the preceding (linearly independent) vectors.

EIGENVECTORS AND EIGENVALUES

- Then there exist scalars $c_1, \ldots, c_p$ such that

$$c_1\mathrm{v}_1 + \cdots + c_p\mathrm{v}_p = \mathrm{v}_{p+1} \qquad \text{----(4)}$$

- Multiplying both sides of (4) by A and using the fact that

$$c_1A\mathrm{v}_1 + \cdots + c_pA\mathrm{v}_p = A\mathrm{v}_{p+1}$$

$$c_1\lambda_1\mathrm{v}_1 + \cdots + c_p\lambda_p\mathrm{v}_p = \lambda_{p+1}\mathrm{v}_{p+1} \qquad \text{----(5)}$$

- Multiplying both sides of (4) by λ_{p+1} and subtracting the result from (5), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathrm{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathrm{v}_p = 0 \qquad \text{----(6)}$$

EIGENVECTORS AND EIGENVALUES

- Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, the weights in (6) are all zero.
- But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct.
- Hence $c_i = 0$ for $i = 1, \ldots, p$.
- But then (4) says that $\mathrm{v}_{p+1} = 0$, which is impossible.

EIGENVECTORS AND DIFFERENCE EQUATIONS

- Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent.

- If A is an $n \times n$ matrix, then

$$\mathrm{x}_{k+1} = A\mathrm{x}_k \qquad (k = 0, 1, 2\ldots) \qquad \text{----(7)}$$

is a *recursive* description of a sequence $\{x_k\}$ in $\mathbb{R}^n$.

- A **solution** of (7) is an explicit description of $\{x_k\}$ whose formula for each x_k does not depend directly on A or on the preceding terms in the sequence other than the initial term $\mathbf{x}_0$.

EIGENVECTORS AND DIFFERENCE EQUATIONS

- The simplest way to build a solution of (7) is to take an eigenvector $\mathbf{x}_0$ and its corresponding eigenvalue λ and let

$$x_k = \lambda^k x_0 \quad (k = 1, 2, \ldots) \qquad \text{----(8)}$$

- This sequence is a solution because

$$Ax_k = A(\lambda^k x_0) = \lambda^k (Ax_0) = \lambda^k (\lambda x_0) = \lambda^{k+1} x_0 = x_{k+1}$$

DOUBLE INTEGRALS

DEFINITE LINE INTEGRAL

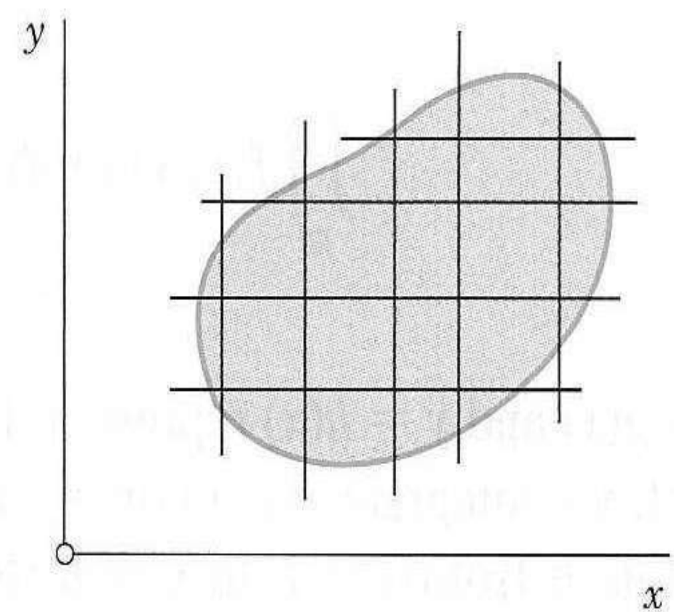
$$\int_a^b f(x)\,dx$$

DOUBLE INTEGRAL

We integrate a function f(x,y),called integrand , over a closed bounded region R in the xy-plane , whose boundary curve has a unique tangent at each point, but may have finitely many cusps (such as vertices of a triangle or rectangle).

- We subdivide the region **R** by drawing parallel to “x” and “y” axes. We number the rectangles that are within **R** from 1 to n. In each such rectangle we choose a point, say, (x_k, y_k) in the ***kth*** rectangle, and then we form the sum

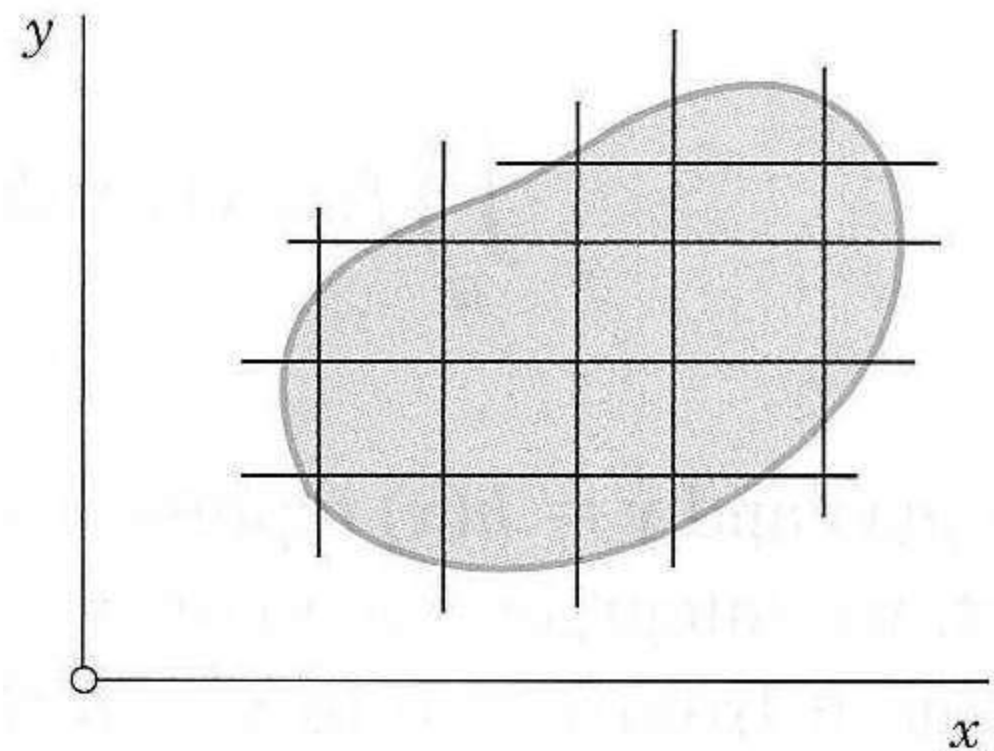
$$J_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$



- Where $\Delta A_k.$ is the area of the ***kth*** rectangle. This we do for larger and larger positive integers n in a completely independent manner but so that the length of the maximum diagonal of the rectangles approaches zero as n approaches infinity.
- In this fashion we obtain a sequence of real numbers
- Assuming $J_{n_1}, J_{n_2}, \cdots$ continuous in **R** and **R** is bounded by finitely many smooth curves , one can show that this sequence converges and its limit is independent of the choice of subdivisions and corresponding points (x_k, y_k).

- This limit is called the ***DOUBLE INTEGRAL*** of f(x,y) over the region **R** and is denoted by

$$\iint_R f(x, y)\, dx\, dy \qquad \text{or} \qquad \iint_R f(x, y)\, dA.$$



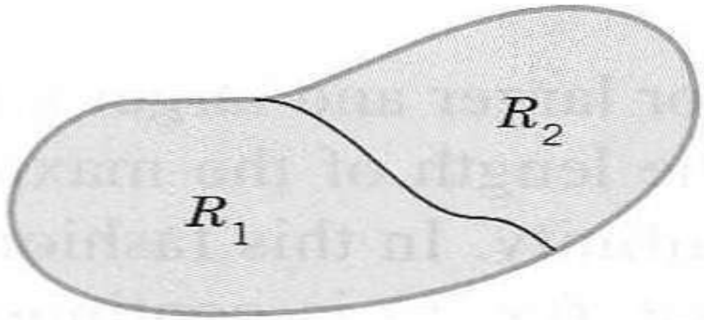
PROPERTIES

- f(x,y) & g(x,y) continuous in a region **R**

$$\iint_R kf\,dx\,dy = k\iint_R f\,dx\,dy$$

$$\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$$

- $$\iint\limits_{R} f \, dx \, dy = \iint\limits_{R_1} f \, dx \, dy + \iint\limits_{R_2} f \, dx \, dy$$



Furthermore, there exists at least one point (x_0, y_0) in **R** such that we have

$$\iint\limits_{R} f(x, y) \, dx \, dy = f(x_0, y_0) A$$

Where <u>*A*</u> is the area of **R;** this is called the ***<u>MEAN VALUE THEOREM</u>*** for double integrals.

EVALUATION OF DOUBLE INTEGRAL

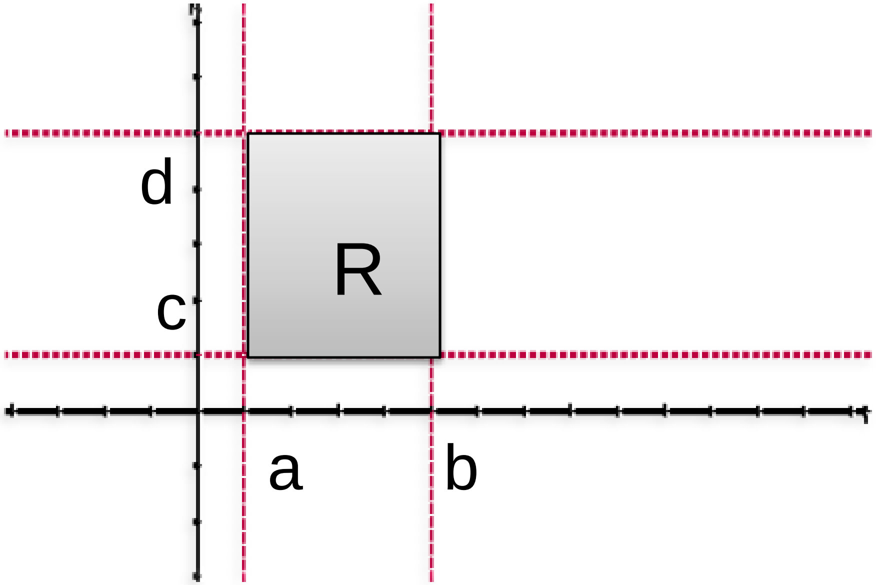
(1) Suppose that **R** can be described by inequalities of the form

$a \leqq x \leqq b \qquad c \leqq y \leqq d$

- represents the boundary of **R.** Then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

•



$$a \leqq x \leqq b$$

$$c \leqq y \leqq d$$

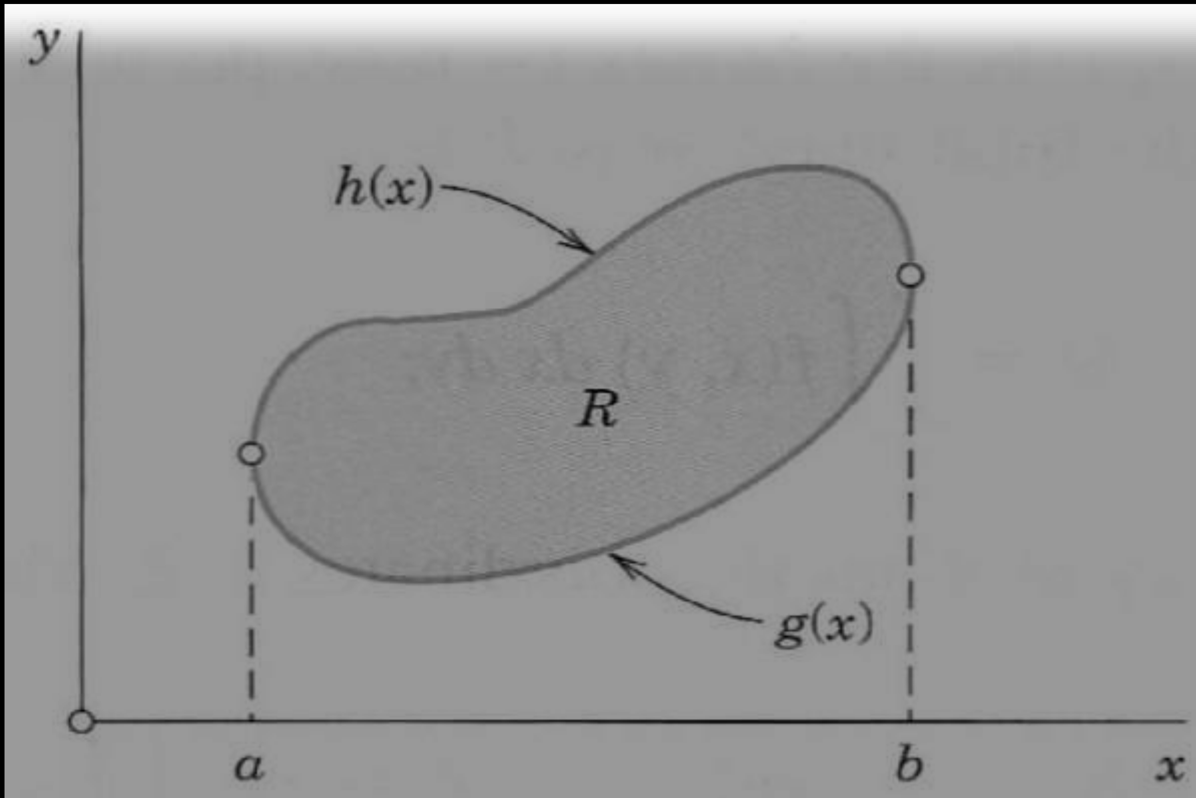
$$\iint\limits_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

(2)Suppose that **R** can be described by inequalities of the form

$$a \leqq x \leqq b \qquad\qquad g(x) \leqq y \leqq h(x)$$

- so that $y = g(x)$ and $y = h(x)$ represents the boundary of **R.** Then

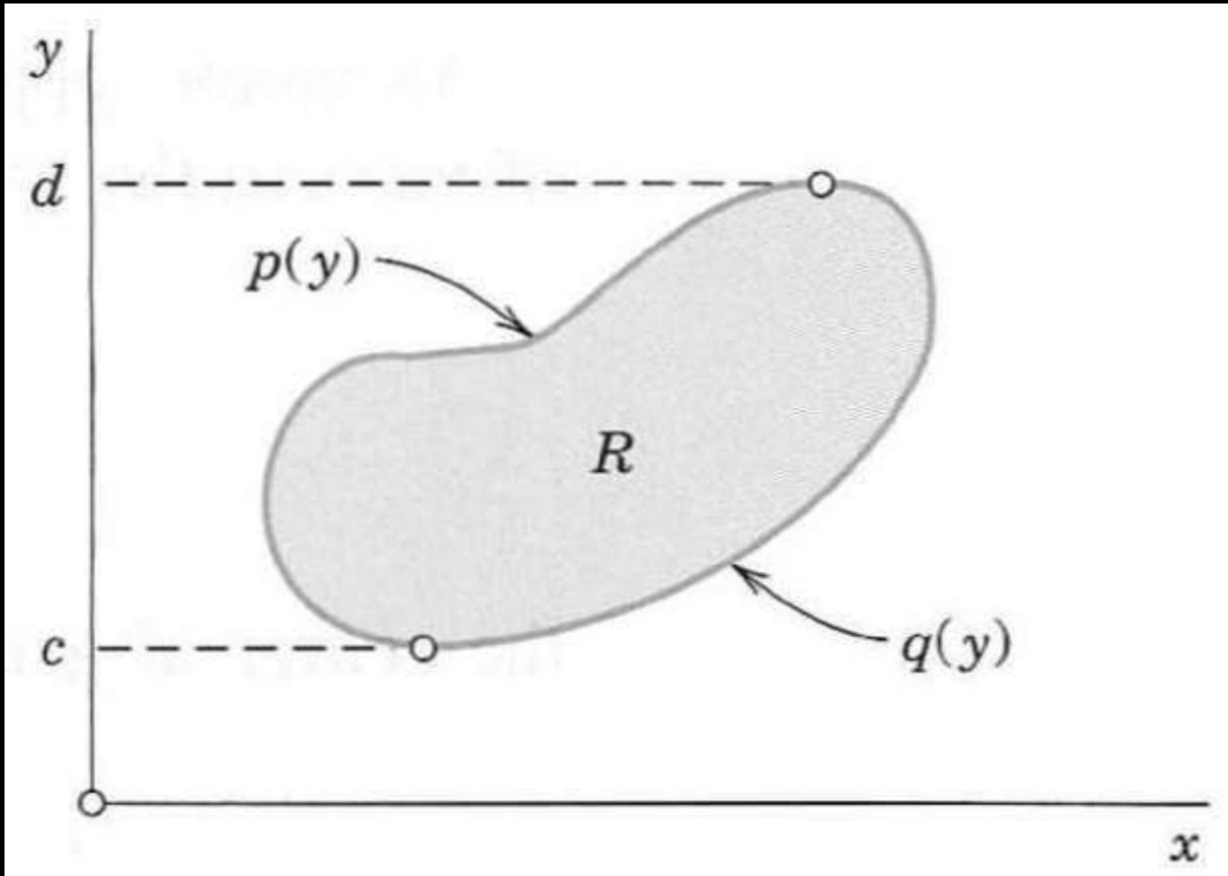
$$\iint_R f(x, y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\,dy\right] dx$$



$$a \leqq x \leqq b$$

$$g(x) \leqq y \leqq h(x)$$

$$\iint_R f(x, y)\, dx\, dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\, dy \right] dx$$



$$c \leqq y \leqq d$$

$$p(y) \leqq x \leqq q(y)$$

$$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x, y)\, dx \right] dy$$

- **NOTE:-** if **R** can not be represented by those inequalities, but can be subdivided into finitely many portions that have that property , we may integrate f(x,y) over each portion separately and add the results; this will give us the value of the double integral of f(x,y) over that region **R.**

APPLICATION OF DOUBLE INTEGRALS

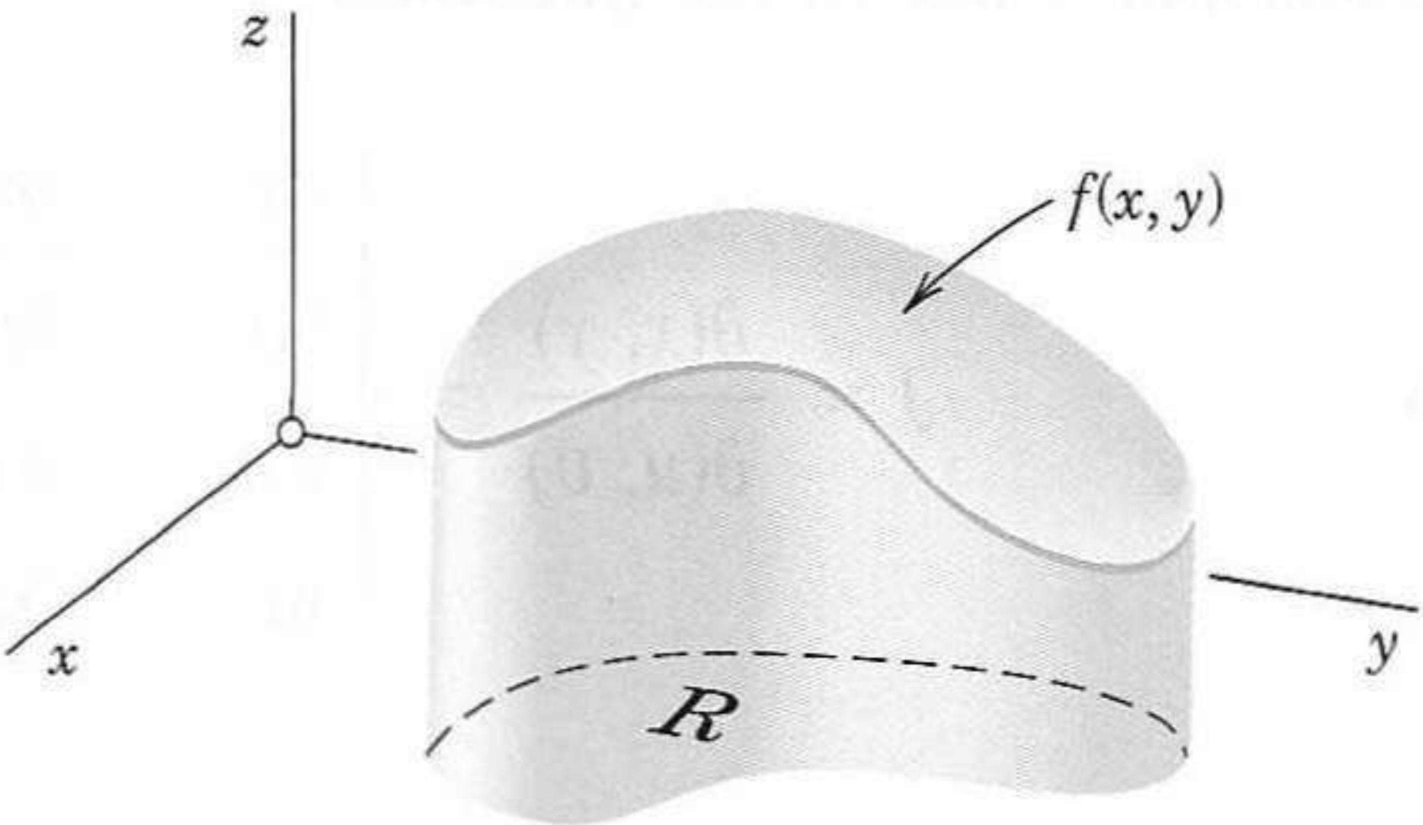
- The **AREA** A of a region **R** in the xy-plane is given by the double integral

$$A = \iint_R dx\, dy$$

- The **VOLUME** V beneath the surface z= f(x,y)>0 and above a region **R** in the xy-plane is

$$V = \iint_R f(x, y)\, dx\, dy$$

- because the term f $(x_k, y_k)\Delta A_k$ in J_n at the beginning of this section represents the volume of a rectangular parallelepiped with base ΔA_k and altitude f (x_k, y_k)



- Let f(x,y) be the density (mass per unit volume) of a distribution of the mass in the xy-plane. Then the **total mass** M in **R** is

$$M = \iint_R f(x, y)\,dx\,dy$$

- The **CENTER OF GRAVITY** of the mass in **R** has the co-ord $\bar{x}, \bar{y}$,e where

$$\bar{x} = \frac{1}{M}\iint_R xf(x, y)\,dx\,dy \quad \& \quad \bar{y} = \frac{1}{M}\iint_R yf(x, y)\,dx\,dy$$

- The ***MOMENT OF INERTIA*** I_x and I_y of the mass in R about the "x" and "y" axis respectively , are

$$I_x = \iint_R y^2 f(x, y)\, dx\, dy, \qquad I_y = \iint_R x^2 f(x, y)\, dx\, dy;$$

- The ***POLAR MOMENT OF INERTIA*** I_0 about the origin of the mass in R is

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) f(x, y)\, dx\, dy.$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(x(u))\,\frac{dx}{du}\,du.$$

- Here assume that x=x(u) is continuous and has a continuous derivative in some interval $\alpha \leqq u \leqq \beta$ such that $x(\alpha) = a,\ x(\beta) = b$ [or $x(\alpha) = b,\ x(\beta) = a$] and x(u) varies between "a" and "b" when "u" varies from α and β.

- The formula for the change of variables in from "x", "y" to "u", "v" is

$$\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv;$$

- that is the integrand is expressed in terms of "u" and "v" , and "dx dy" is replaced by "du dv" times the absolute value of the ***JACOBIAN***.

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- Here we assume the following. The functions $x=x(u,v) \quad y=y(u,v)$
- effecting the change are continuous and have continuous partial derivatives in some region R* in the uv-plane such that the point (x,y) corresponding to any (u,v) in R* lies in r and, conversely , to every (x,y) in Rthere corresponds one and only one (u,v) in R*; furthermore the **JACOBIAN** **J** is either positive throughout R* or negative throughout R*.

- For polar co-ordinate “r” and “θ”
- $x = r\cos\theta$ and $y = r\sin\theta$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

- where R* is the region in the rθ-plane corresponding to R in the xy-plane.