SPRINGS

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Mechanics of Materials

by EJ Hearn

OPEN COIL SPRINGS

- Springs are energy-absorbing units whose function it is to store energy and to release it slowly or rapidly depending on the particular application.
- In motor vehicle applications the springs act as buffers between the vehicle itself and the external forces applied through the wheels by uneven road conditions.
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- In such cases the shock loads are converted into strain energy of the spring and the resulting effect on the vehicle body is much reduced.
- In some cases springs are merely used as positioning devices whose function it is to return mechanisms to their original positions after some external force has been removed.
- From a design point of view “good springs store and release energy but do not significantly absorb it. Should they do so then they will be prone to failure.

Open-coiled helical spring subjected to axial load *W*

(a) Defection

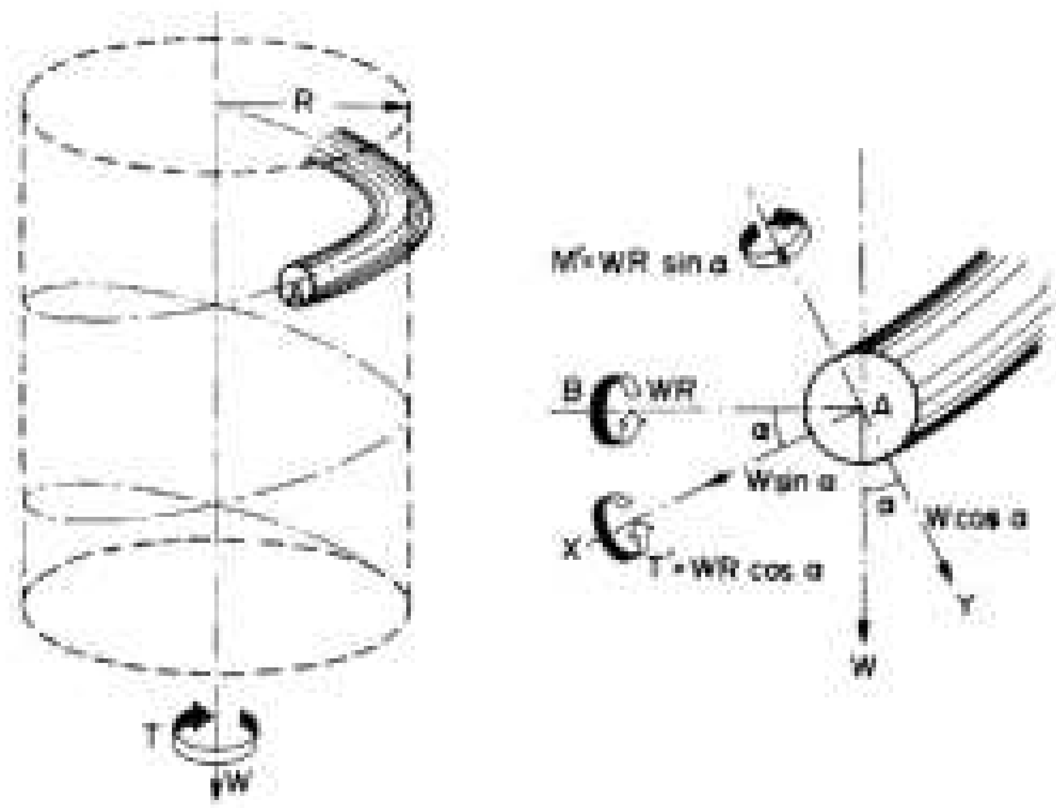


Fig. 12.3. Open-coiled helical spring.

- In an open coiled spring the coils are no longer so close together that the effect of the helix angle *a can be neglected and the spring is subjected to comparable bending and twisting* effects.

- The axial load W can now be considered as a direct load W acting on the spring at the mean radius R, together with a couple WR about AB

- *This couple has a* component about AX of WR cos *a tending to twist the section, and a component about AY* of WR sin *a tending to reduce the curvature of the coils, i.e. a bending effect.*

- The shearing effect of W across the spring section is neglected as being very small in comparison with the other effects.

Thus $$T' = WR\cos\alpha \quad \text{and} \quad M' = WR\sin\alpha$$

Now, the total strain energy, neglecting shear,

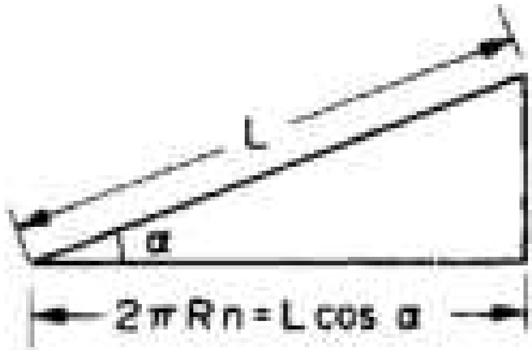
$$U = \frac{T^2 L}{2GJ} + \frac{M^2 L}{2EI} \quad \text{(see §§ 11.3 and 11.4)}$$

$$= \frac{L(WR\cos\alpha)^2}{2GJ} + \frac{L(WR\sin\alpha)^2}{2EI}$$

$$= \frac{LW^2R^2}{2}\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

and this must equal the total work done $\frac{1}{2}W\delta$.

$$\therefore \qquad \frac{1}{2}W\delta = \frac{LW^2R^2}{2}\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$



From the helix form of Fig. 12.4

$$2\pi Rn = L\cos\alpha$$

$$\therefore \qquad L = 2\pi Rn\sec\alpha$$

$$\therefore \qquad \textbf{deflection } \delta = 2\pi n\, WR^3\sec\alpha\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

$$\text{stiffness } S = \frac{W}{\delta}$$

$$\frac{1}{S} = \frac{\delta}{W} = 2\pi n R^3 \sec \alpha \left[\frac{\cos^2 \alpha}{GJ} + \frac{\sin^2 \alpha}{EI} \right]$$

(*b*) *Maximum stress*

$$\sigma = \frac{My}{I} \quad \text{with} \quad M = WR \sin \alpha$$

$$\tau = \frac{Tr}{J} \quad \text{with} \quad T = WR \cos \alpha$$

The principal stresses at any point can then be obtained analytically or graphically

(c) *Angular rotation*

Consider an imaginary axial torque T applied to the spring, together with W producing an angular rotation θ *of one end of the spring relative to the other*.

The combined twisting moment on the spring cross-section is then

$$\bar{T} = WR\cos\alpha + T\sin\alpha$$

and the combined bending moment

$$\bar{M} = T\cos\alpha - WR\sin\alpha$$

The total strain energy of the system is then

$$U = \frac{\bar{T}^2 L}{2GJ} + \frac{\bar{M}^2 L}{2EI}$$

$$= \frac{(WR\cos\alpha + T\sin\alpha)^2 L}{2GJ} + \frac{(T\cos\alpha - WR\sin\alpha)^2 L}{2EI}$$

Castigliano's theorem the angle of twist in the direction of the axial torque T is

given by $\theta = \dfrac{\partial U}{\partial T}$ and since $T = 0$ all terms including T may be ignored.

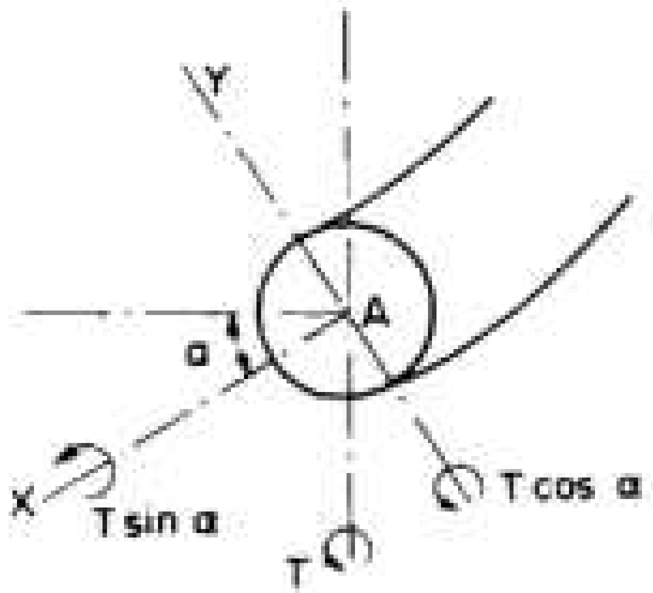
$$\theta = \frac{2WR\cos\alpha\sin\alpha\, L}{2GJ} + \frac{(-2WR\sin\alpha\cos\alpha)\, L}{2EI}$$

$$= WRL\cos\alpha\sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

$$\theta = 2\pi n WR^2\sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

Open-coiled helical spring subjected to axial torque *T*

(a) Wind-up angle



Open-coiled helical spring subjected to axial torque *T*.

Torsional component T sin ***α about AX***

Flexural (bending) component T cos ***α about AY tending to increase the curvature of the coils.***

$$\text{strain energy } U = \frac{T^2 L}{2GJ} + \frac{M^2 L}{2EI}$$

$$= \frac{L}{2}\left[\frac{(T \sin \alpha)^2}{GJ} + \frac{(T \cos \alpha)^2}{EI}\right]$$

$$= \frac{T^2 L}{2}\left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI}\right]$$

and this is equal to the work done by T, namely, $\frac{1}{2}T\theta$, where θ is the angle turned through by one end relative to the other, i.e. the wind-up angle of the spring.

$$\therefore \qquad \tfrac{1}{2}T\theta = \tfrac{1}{2}T^2 L\left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI}\right]$$

and, with $L = 2\pi R n \sec \alpha$ as before,

$$\textbf{wind-up angle } \theta = 2\pi n R T \sec \alpha \left[\frac{\sin^2 \alpha}{GJ} + \frac{\cos^2 \alpha}{EI}\right] \qquad (12.11)$$

(b) *Maximum stress*

$$\sigma = \frac{My}{I} \quad \text{with} \quad M = T\cos\alpha$$

$$\tau = \frac{Tr}{J} \quad \text{with} \quad T = T\sin\alpha$$

The principal stresses at any point can then be obtained analytically or graphically

(c) *Axial deflection*

$$U = \frac{T^2 L}{2GJ} + \frac{M^2 L}{2EI}$$

Assuming an imaginary axial load W applied to the spring the total strain energy is given by eqn.

$$U = \frac{(WR\cos\alpha + T\sin\alpha)^2 L}{2GJ} + \frac{(T\cos\alpha - WR\sin\alpha)^2 L}{2EI}$$

Castigliano's theorem the deflection in the direction of W is given by

$$\delta = \frac{\partial U}{\partial W}$$

$$= TRL\cos\alpha\sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right] \quad \text{when } W = 0$$

$$\textbf{deflection } \delta = 2\pi n T R^2 \sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

Leaf or carriage spring: semi-elliptic

The principle of using a beam in bending as a spring has been known for many years and widely used in motor-vehicle applications.

If the beam is arranged as a simple cantilever, as in Fig. 12.7a, it is called a ***quarter-elliptic spring, and***

if as a simply supported beam with central load, as in Fig. 12.7b, it **is termed a *half or semi-elliptic spring.***

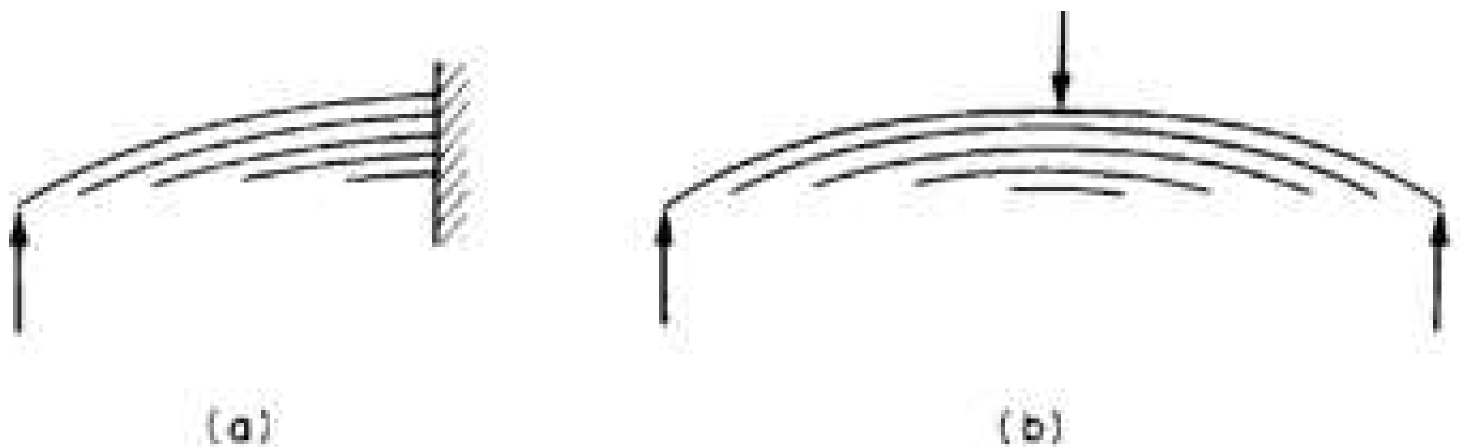


Fig. 12.7. (a) Quarter-elliptic, (b) semi-elliptic, carriage springs.

•Consider the semi-elliptic leaf spring shown in Fig. 12.8.

• With a constant thickness '***t'*** ***this*** design of spring gives a uniform stress throughout and is therefore economical in both material and weight.

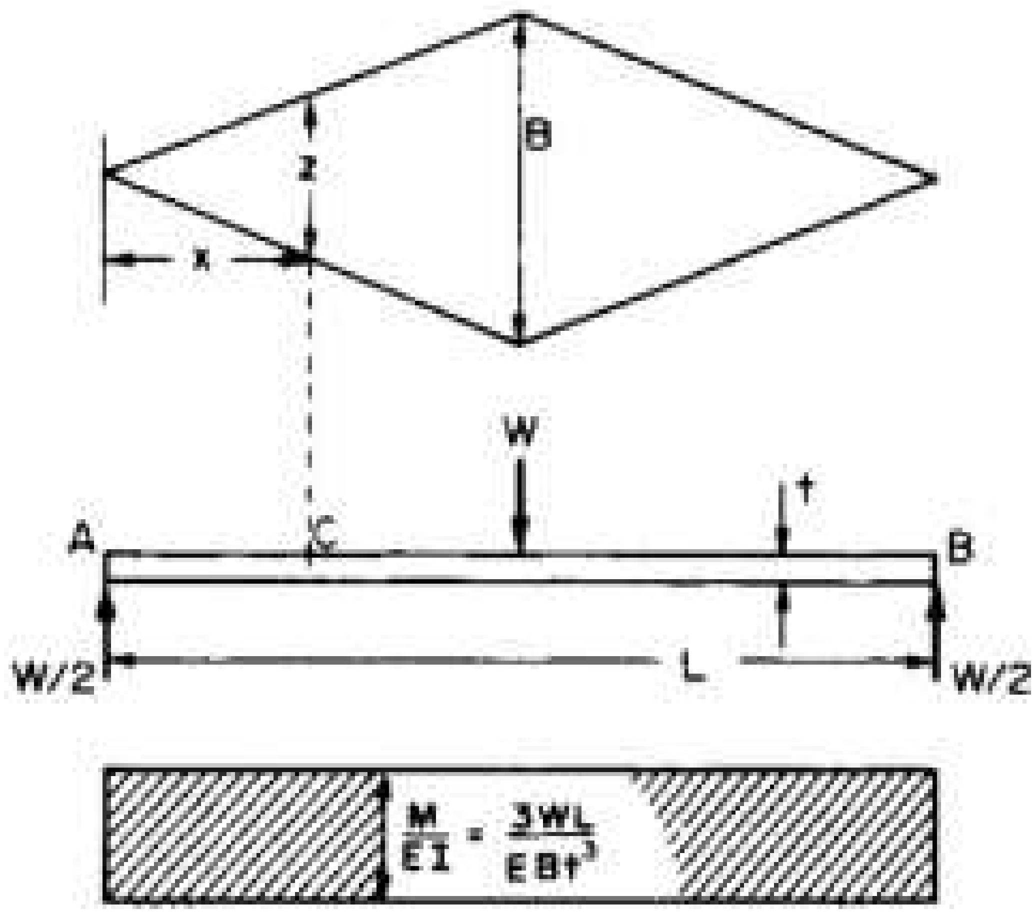


Fig. 12.8. Semi-elliptic leaf spring.

By proportions $\frac{z}{x} = \frac{B}{L/2} \quad \therefore\ z = \frac{2Bx}{L}$

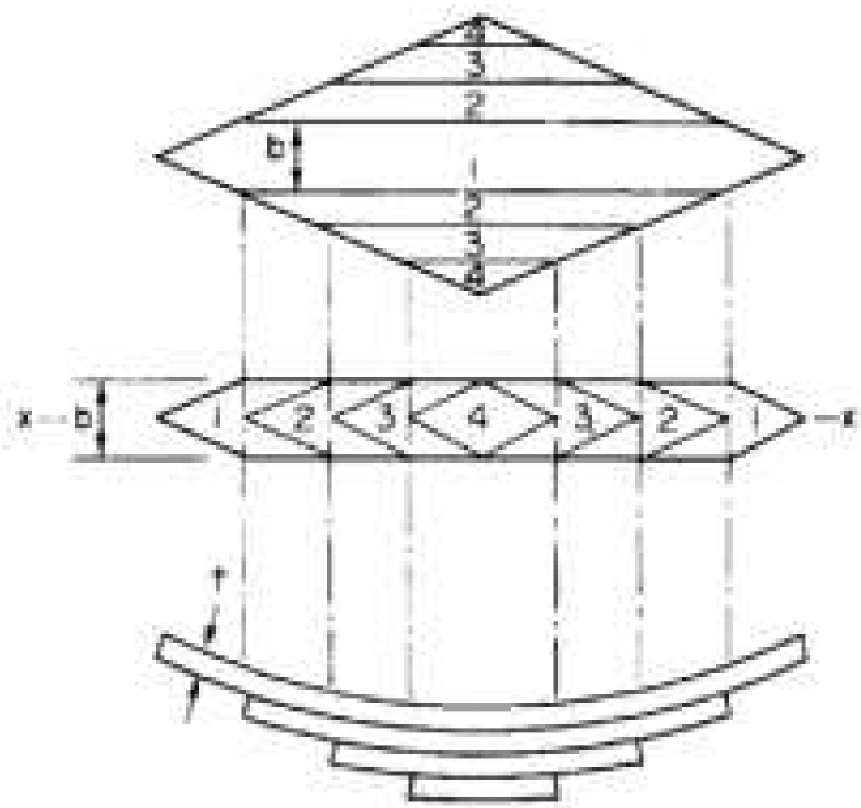
Bending moment at $C = \frac{Wx}{2}$ and $I = \frac{zt^3}{12} = \frac{2Bxt^3}{12L}$

$$\sigma = \frac{My}{I} = \frac{Wx}{2} \times \frac{t}{2} \times \frac{12L}{2Bxt^3}$$

$$= \frac{3WL}{2Bt^2}$$

i.e. the bending stress in a semi-elliptic leaf spring is independent of **x**

If the spring is constructed from strips and placed one on top of the other as shown in Fig. ***12.9, uniform stress conditions are retained, since if the strips are cut along XX and*** replaced side by side, the equivalent leaf spring is obtained as shown.



). Semi-elliptic carriage spring showing initial pre-forming.

Such a spring is then termed a ***carriage spring with n strips of width b, i.e. B = nb.***

Therefore the bending stress in a semi-elliptic carriage spring is

$$\frac{3WL}{2nbt^2}$$

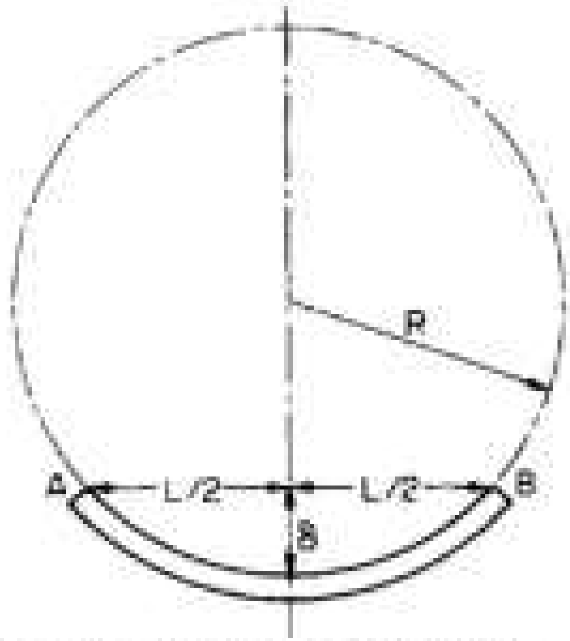
(b) Deflection

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \qquad \therefore\ R = \frac{EI}{M}$$

$$R = E \times \frac{2Bxt^3}{12L} \times \frac{2}{Wx} = \frac{EBt^3}{3WL}$$

i.e. for a given spring and given load, ***R is constant and the spring bends into the arc of a circle.***



From the properties of intersecting chords

$$\delta(2R-\delta) = \frac{L}{2} \times \frac{L}{2}$$

Neglecting δ^2 as the product of small quantities

$$\delta = \frac{L^2}{8R}$$

$$= \frac{L^2}{8} \times \frac{3WL}{EBt^3}$$

i.e. deflection of a semi-elliptic *leaf* spring

$$\delta = \frac{3WL^3}{8EBt^3}$$

But $B = nb$, so that the deflection of a semi-elliptic *carriage* spring is given by

$$\delta = \frac{3WL^3}{8Enbt^3}$$

(c) *Proof load*

The proof load of a leaf or carriage spring is the load which is required to straighten the plates from their initial preformed position.

The maximum bending stress any given load W is

$$\sigma = \frac{3WL}{2nbt^2}$$

Thus if σ_p denotes the stress corresponding to the application of the proof load W_p

$$W_p = \frac{2nbt^2}{3L}\sigma_p$$

Now from eqn.

$$R = \frac{EBt^3}{3WL}$$

and inserting B = nb,

the load W which would produce bending of a flat carriage spring to some radius ***R is given by***

$$W = \frac{Enbt^3}{3RL}$$

Conversely, therefore, the load which is required to straighten a spring from radius ***R will be*** of the same value,

$$W_p = \frac{Enbt^3}{3RL}$$

Substituting for $R = \frac{L^2}{8\delta}$

$$\text{proof load } W_p = \frac{8Enbt^3}{3L^3}\delta_p$$

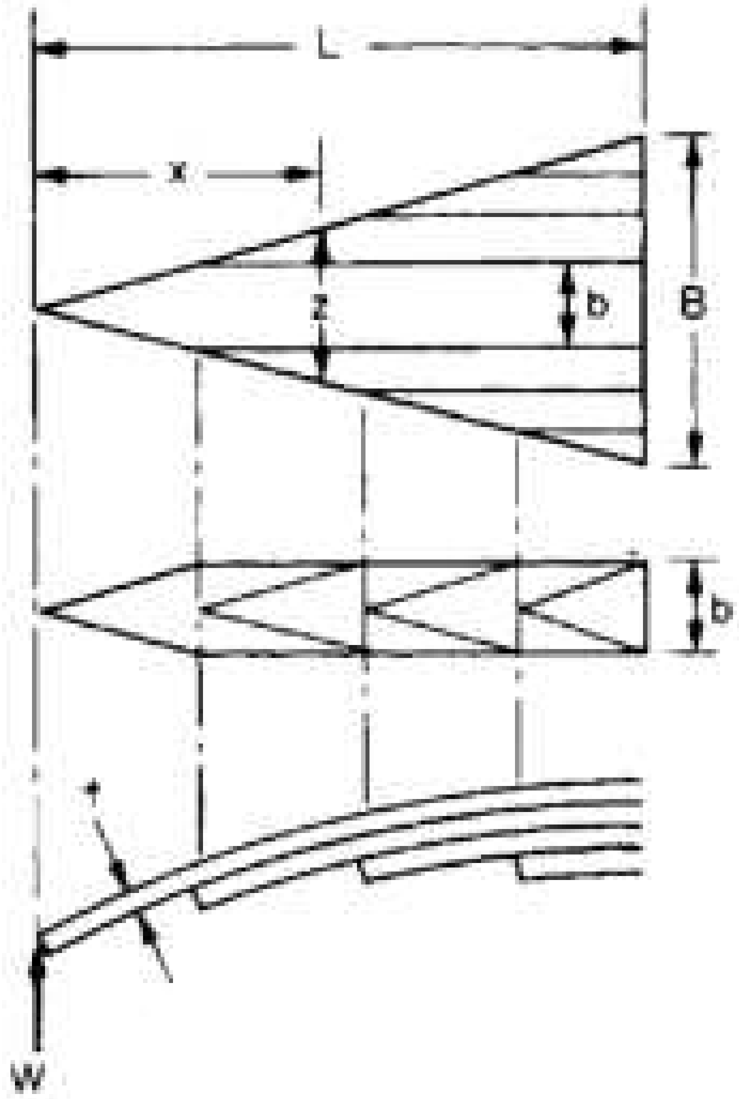
where δ_p is the initial central "deflection" of the spring.

$$\frac{2nbt^2}{3L}\sigma_p = \frac{8Enbt^3}{3L^3}\delta_p$$

$$\text{proof stress } \sigma_p = \frac{4tE}{L^2}\delta_p$$

The above equation therefore yields the correct relationship between the thickness and initial curvature of the spring plates.

Leaf or carriage spring: quarter-elliptic



$$z = \frac{Bx}{L} \quad \text{and} \quad \text{B.M. at } C = Wx$$

$$I = \frac{zt^3}{12} = \frac{Bxt^3}{12L}$$

$$\sigma = \frac{My}{I} = \frac{Wxt}{2} \times \frac{12L}{Bxt^3} = \frac{6WL}{Bt^2}$$

Therefore the maximum bending stress for a quarter-elliptic *leaf* spring

$$= \frac{6WL}{Bt^2}$$

and the maximum bending stress for a quarter-elliptic *carriage* spring

$$= \frac{6WL}{nbt^2}$$

(*b*) *Deflection*

With B.M. at C = Wx and replacing $L/2$ by L in the proof of §12.7(b),

$$\delta = \frac{L^2}{2R}$$

and

$$R = \frac{EI}{M} = \frac{E}{Wx} \times \frac{Bxt^3}{12L} = \frac{Ebt^3}{12WL}$$

$$\delta = \frac{L^2}{2} \times \frac{12WL}{EBt^3} = \frac{6WL^3}{EBt^3}$$

Therefore deflection of a quarter-elliptic *leaf* spring

$$= \frac{6WL^3}{EBt^3}$$

and deflection of a quarter-elliptic *carriage* spring

$$= \frac{6WL^3}{Enbt^3}$$

Spiral spring

(a)Wind-up angle

•Spiral springs are normally constructed from thin rectangular-section strips wound into a spiral in one plane.

•They are often used in clockwork mechanisms, the winding torque or

•moment being applied to the central spindle and the other end firmly anchored to a pin at the outside of the spiral.

• Under the action of this central moment all sections of the spring will be subjected to uniform bending which tends to reduce the radius of curvature at all points.

Consider now the spiral spring shown in Fig.

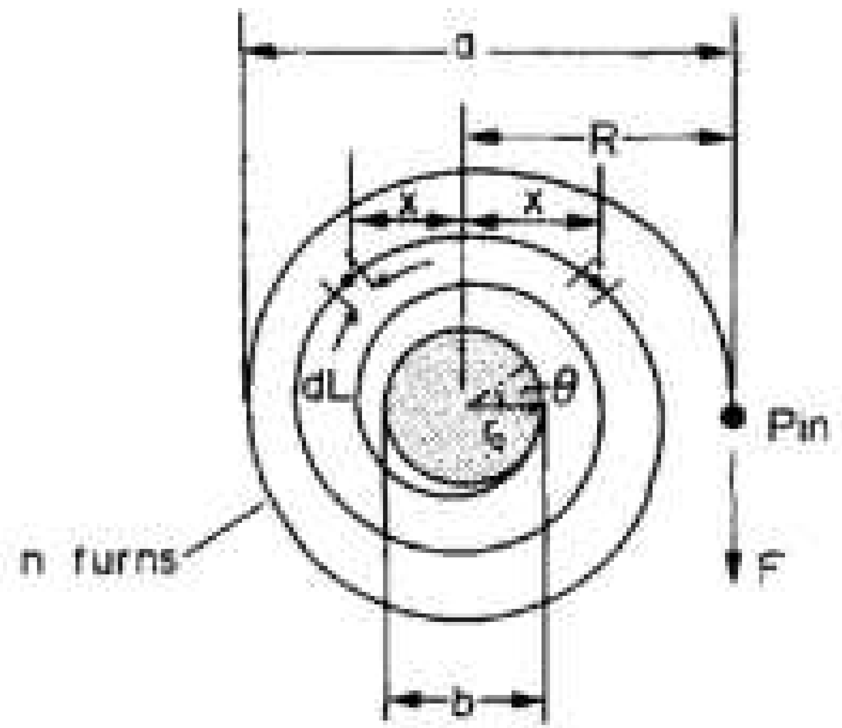


Fig. 12.12. Spiral spring.

Let M = winding moment applied to the spring spindle,

R = radius of spring from spindle to pin,

a = maximum dimension of the spring from the pin,

B = breadth of the material of the spring,

t = thickness of the material of the spring,

b = diameter of the spindle.

Assuming the polar equation of the spiral to be that of an Archimedean spiral,

$$r = r_0 + \left(\frac{A}{2\pi}\right)\theta$$

where A is some constant

When $\theta = 0, \quad r = r_0 = \frac{b}{2}$

and for the nth turn, $\theta = 2n\pi$ and

$$r = \frac{a}{2} = \frac{b}{2} + \left(\frac{A}{2\pi}\right)2n\pi$$

$$\therefore \qquad A = \frac{(a-b)}{2n}$$

i.e. the equation to the spiral is

$$r = \frac{b}{2} + \frac{(a-b)}{4\pi n}\theta$$

When a torque or winding couple M is applied to the spindle a resistive force ***F will be set*** up at the pin such that

$$\text{winding couple } M = F \times R$$

Consider now two small elements of material of length ***dl at distance x to each side of the*** centre line (Fig. 12.12).

For small deflections, from Mohr's area-moment method the change in slope between two points is

$$\left(\frac{M}{EI}\right) dL$$

For the portion on the left,

$$\text{change in slope} = d\theta_1 = \frac{F(R+x)\,dL}{EI}$$

and similarly for the right-hand portion,

$$\text{change in slope} = d\theta_2 = \frac{F(R-x)\,dL}{EI}$$

The sum of these changes in slope is thus

$$d\theta_1 + d\theta_2 = \frac{F(R+x)\,dL}{EI} + \frac{F(R-x)\,dL}{EI}$$

$$= \frac{2FR\,dL}{EI}$$

If this is integrated along the length of the spring the result obtained will be twice the total change in slope along the spring, i.e. twice the angle of twist.

$$\text{angle of twist} = \tfrac{1}{2}\int_0^L \frac{2FR\,dL}{EI} = \frac{FRL}{EI} = \frac{ML}{EI}$$

$$L = \int_0^L dL = \int_0^{2n\pi} r d\theta = \int_0^{2n\pi} \frac{b}{2} + \frac{(a-b)}{4\pi n}\theta \, d\theta$$

$$= \left[\frac{b\theta}{2} + \frac{(a-b)}{4\pi n}\frac{\theta^2}{2}\right]_0^{2n\pi} = \left[\frac{2nb\pi}{2} + \frac{(a-b)}{4\pi n}\frac{(2n\pi)^2}{2}\right]$$

$$= \pi n\left[b + \frac{(a-b)}{2}\right]$$

$$= \frac{\pi n}{2}\left[a+b\right]$$

Therefore the wind-up angle of a spiral spring is

$$\theta = \frac{M}{EI}\left[\frac{\pi n}{2}(a+b)\right]$$

(b) *Maximum stress*

The maximum bending stress set up in the spring will be at the point of greatest bending moment, since the material of the spring is subjected to pure bending.

$$\text{Maximum bending moment} = F \times a$$

$$\text{maximum bending stress} = \frac{My}{I} = \frac{Fa(t/2)}{I}$$

But, for rectangular-section spring material of breadth ***B and thickness t,***

$$I = \frac{Bt^3}{12}$$

$$\sigma_{\max} = \frac{Fat}{2} \times \frac{12}{Bt^3} = \frac{6Fa}{Bt^2}$$

STRAIN ENERGY

REPRODUCED MAJORLY FROM MECHANICS OF MATERIALS

BY

EJ HEARN

- Energy is normally defined as the *capacity to do work and it may exist in any of many forms,* e.g. mechanical (potential or kinetic), thermal, nuclear, chemical, etc.

- The potential energy of a body is the form of energy which is stored by virtue of the work which has previously been done on that body, e.g. in lifting it to some height above a datum.

- Strain energy is a particular form of potential energy which is stored within materials which have been subjected to strain, i.e. to some change in dimension.

- The material is then capable of doing work, equivalent to the amount of strain energy stored, when it returns to its original unstrained dimension.

- *Strain energy is therefore defined as the energy which is stored within a material when work has been done on the material.*

- *Here it is assumed that the material remains elastic whilst work* is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

i.e. strain energy ***U = work done***

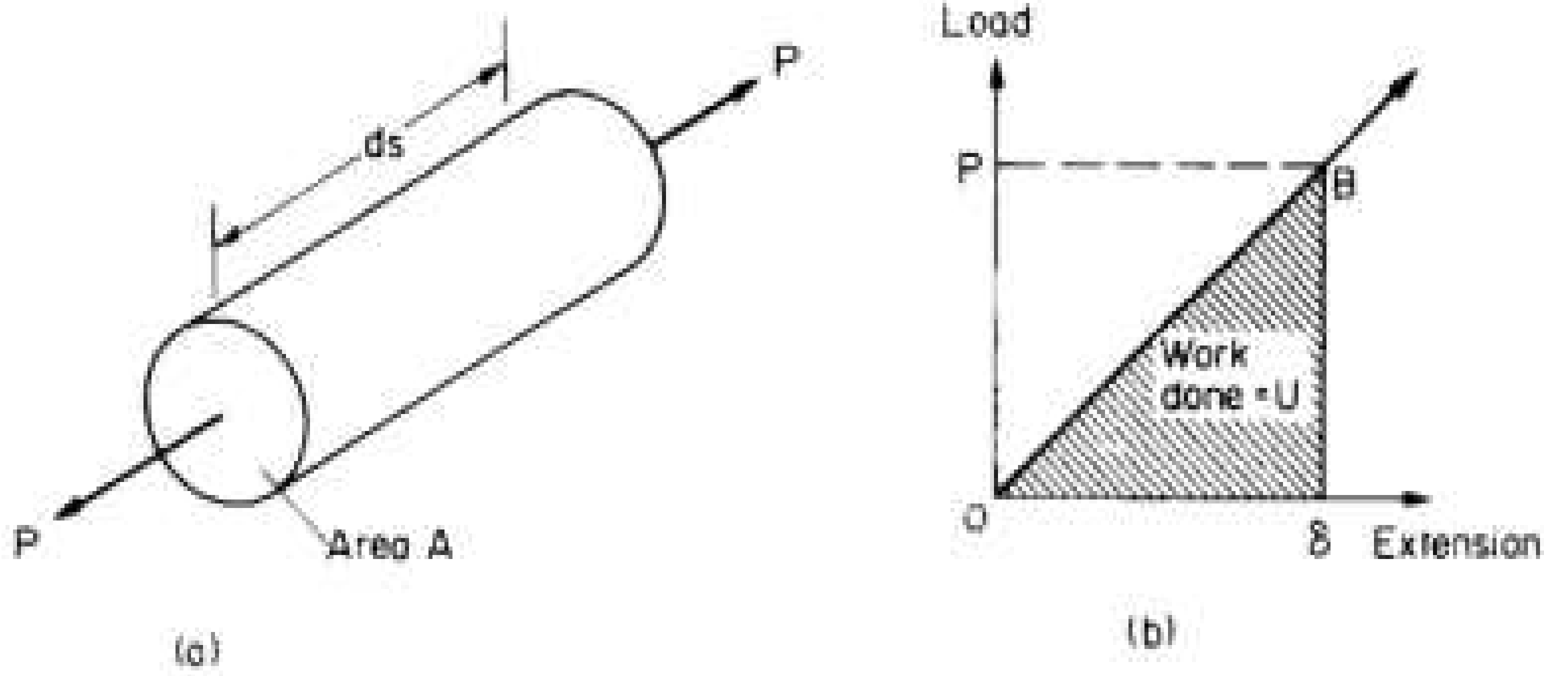


Fig. 11.1. Work done by a gradually applied load.

Thus for a gradually applied load the work done in straining the material will be given by

$$U = \tfrac{1}{2} P\delta$$

- The strain energy per unit volume is often referred to as the ***resilience.***

- ***The value of the*** resilience at the yield point or at the proof stress for non-ferrous materials is then termed the ***proof resilience***

Strain energy - tension or compression

$$\text{strain energy } U = \tfrac{1}{2} P\delta$$

$$\text{Young's modulus } E = \frac{\text{stress}}{\text{strain}} = \frac{P}{A} \times \frac{ds}{\delta}$$

$$\delta = \frac{P\,ds}{AE}$$

$$\text{for the bar element } U = \frac{P^2\,ds}{2AE}$$

$$\therefore \quad \text{total strain energy for a bar of length } L = \int_0^L \frac{P^2\,ds}{2AE}$$

Thus, assuming that the area of the bar remains constant along the length,

$$U = \frac{P^2 L}{2AE}$$

or, in terms of the stress $\sigma\,(= P/A)$,

$$U = \frac{\sigma^2 AL}{2E} = \frac{\sigma^2}{2E} \times \textbf{volume of bar} \tag{11.2}$$

i.e. strain energy, or resilience, *per unit volume* of a bar subjected to direct load, tensile or compressive

$$= \frac{\sigma^2}{2E} \tag{11.3}$$

or, alternatively,

$$= \frac{1}{2}\sigma \times \frac{\sigma}{E} = \tfrac{1}{2}\sigma \times \varepsilon$$

i.e. **resilience = $\tfrac{1}{2}$ stress × strain**

INCLUDING THE WEIGHT OF THE BAR

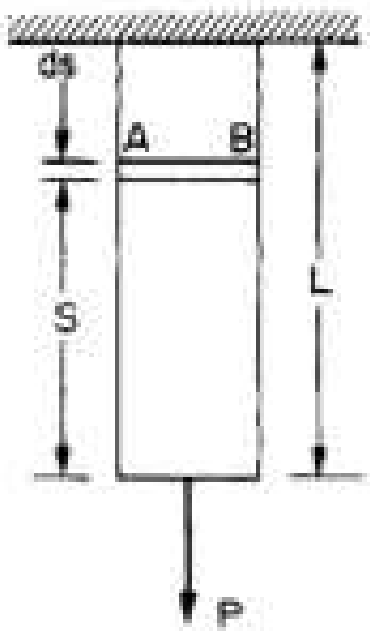


Fig. 11.2. Direct load – tension or compression.

Assuming a uniform cross-section of area A with density ρ,

$$\text{load on section } AB = P \pm \rho g A s$$

Thus, for a tensile force P the extension of the element ds is given by the definition of Young's modulus E to be

$$\delta = \frac{\sigma ds}{E}$$

$$= \frac{(P + \delta g As)}{AE} ds$$

$$\text{work done} = \tfrac{1}{2} \times \text{load} \times \text{extension}$$

$$= \tfrac{1}{2}(P + \rho g As)\frac{(P + \rho g As)}{AE} ds$$

$$= \frac{P^2}{2AE} ds + \frac{P\rho g}{E} s\, ds + \frac{(\rho g)^2 A}{2E} s^2\, ds$$

$\therefore$ total strain energy or work done

$$= \int_0^L \frac{P^2}{2AE}\,ds + \int_0^L \frac{P\rho g}{E}\,s\,ds + \int_0^L \frac{(\rho g)^2 A}{2E}\,s^2\,ds$$

$$= \frac{P^2 L}{2AE} + \frac{P\rho g L^2}{2E} + \frac{(\rho g)^2 A L^3}{6E}$$

The last two terms are therefore the modifying terms to eqn. (11.1) to account for the body-weight effect of the bar.

Strain energy – shear

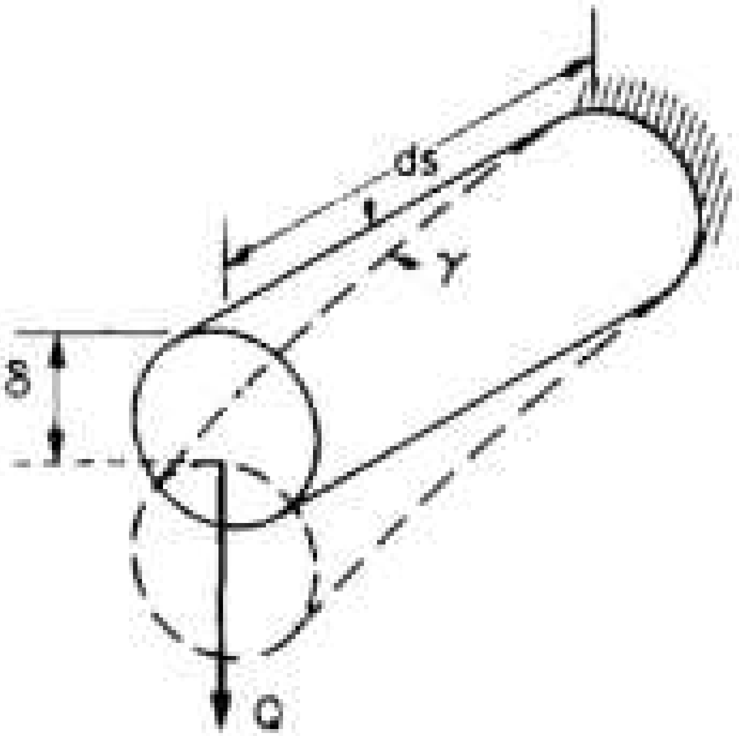


Fig. 11.3. Shear.

Strain energy U = work done $= \frac{1}{2}Q\delta = \frac{1}{2}Q\gamma\, ds$

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$$

$$\gamma = \frac{Q}{AG}$$

$$\text{shear strain energy} = \tfrac{1}{2} Q \times \frac{Q}{AG} \times ds = \frac{Q^2}{2AG} ds$$

$$\therefore \text{ total strain energy resulting from shear } = \int_0^L \frac{Q^2 ds}{2AG} = \frac{Q^2 L}{2AG}$$

in terms of the shear stress $\tau = (Q/A)$,

$$U = \frac{\tau^2 A L}{2G} = \frac{\tau^2}{2G} \times \text{volume of bar}$$

Strain energy – bending

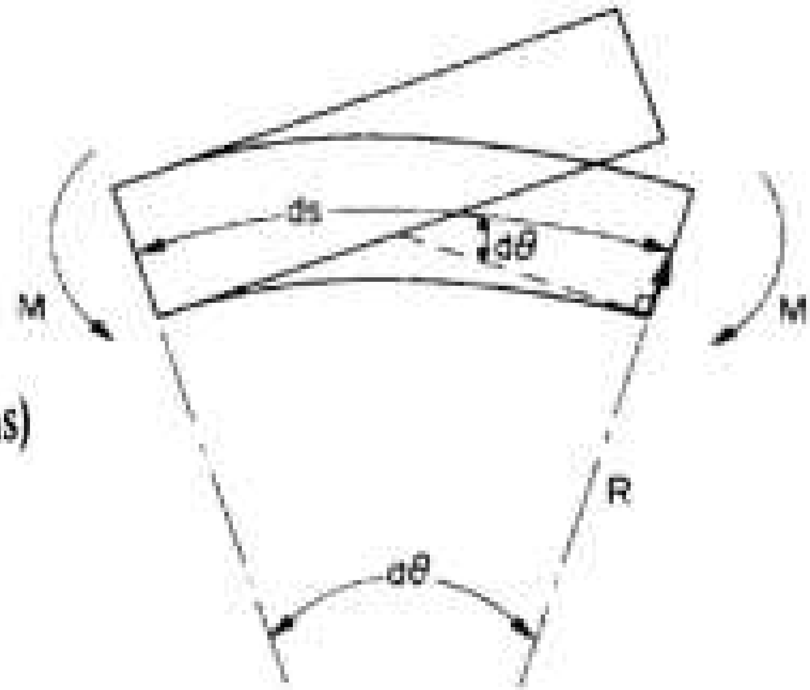


Fig. 11.4. Bending.

Strain energy = work done

$= \frac{1}{2} \times$ moment $\times$ angle turned through (in radians)

$= \frac{1}{2} M\, d\theta$

But $\quad ds = R\, d\theta \quad$ and $\quad \frac{M}{I} = \frac{E}{R}$

$$\therefore \quad d\theta = \frac{ds}{R} = \frac{M}{EI} ds$$

$$\therefore \quad \text{strain energy} = \frac{1}{2} M \times \frac{M}{EI} ds = \frac{M^2 ds}{2EI}$$

Total strain energy resulting from bending,

$$U = \int_0^L \frac{M^2\,ds}{2EI}$$

If the bending moment is constant this reduces to

$$U = \frac{M^2 L}{2EI}$$

Strain energy – torsion

Strain energy = work done = $\frac{1}{2}T\,d\theta$

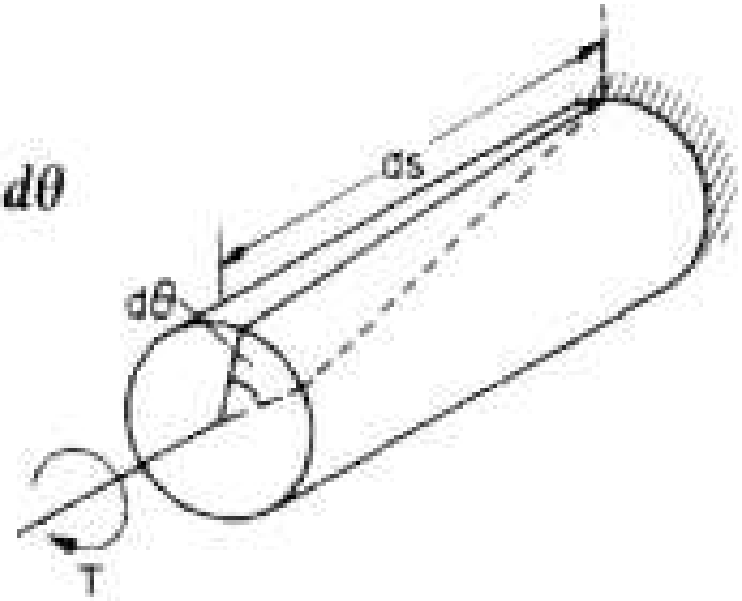


Fig. 11.5. Torsion.

But, from the simple torsion theory,

$$\frac{T}{J} = \frac{G\,d\theta}{ds} \quad \text{and} \quad d\theta = \frac{T\,ds}{GJ}$$

total strain energy resulting from torsion,

$$U = \int_0^L \frac{T^2\,ds}{2GJ} = \frac{T^2 L}{2GJ}$$

since in most practical applications T is constant.

For a hollow circular shaft eqn. (11.8) still applies

i.e. Strain energy $U = \dfrac{T^2 L}{2GJ}$

$$\frac{T}{J} = \frac{\tau}{r} = \frac{\tau_{max}}{R}$$

$$J = \frac{\pi}{2}\left(R^4 - r^4\right)$$

$$T = \frac{\pi}{2R}\,\tau_{max}\left(R^4 - r^4\right)$$

$$U = \frac{\left[\frac{\pi\tau_{max}}{2R}(R^4 - r^4)\right]^2 L}{2G\frac{\pi}{2}(R^4 - r^4)}$$

$$= \frac{\tau_{max}^2}{4G}\frac{\pi(R^4 - r^4)L}{R^2}$$

$$= \frac{\tau_{max}^2}{4G}\frac{[R^2 + r^2]}{R^2} \times \text{volume of shaft}$$

Strain energy/unit volume $= \frac{\tau_{max}^2}{4G}\frac{[R^2 + r^2]}{R^2}$

It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

i.e.

$$\text{strain energy } U = \frac{(\text{applied "load"})^2 \times L}{2 \times \text{product of two related constants}}$$

the constants being related to the type of loading considered. In bending, for example, the relevant constants which appear in the bending theory are E and I, whilst for torsion G and J are more applicable. Thus the above standard equations for strain energy should easily be remembered.

Strain energy of three-dimensional stress system

Total strain energy

$$U_t = \Sigma \tfrac{1}{2}\sigma\varepsilon$$

$$U_t = \tfrac{1}{2}\sigma_1\varepsilon_1 + \tfrac{1}{2}\sigma_2\varepsilon_2 + \tfrac{1}{2}\sigma_3\varepsilon_3$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z)$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$$

$$U_t = \frac{1}{2E}[\sigma_1(\sigma_1 - v\sigma_2 - v\sigma_3) + \sigma_2(\sigma_2 - v\sigma_3 - v\sigma_1) + \sigma_3(\sigma_3 - v\sigma_2 - v\sigma_1)]$$

$$U_t = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \text{ per unit volume}$$

Equation 14.21

Shear (or "distortion") strain energy

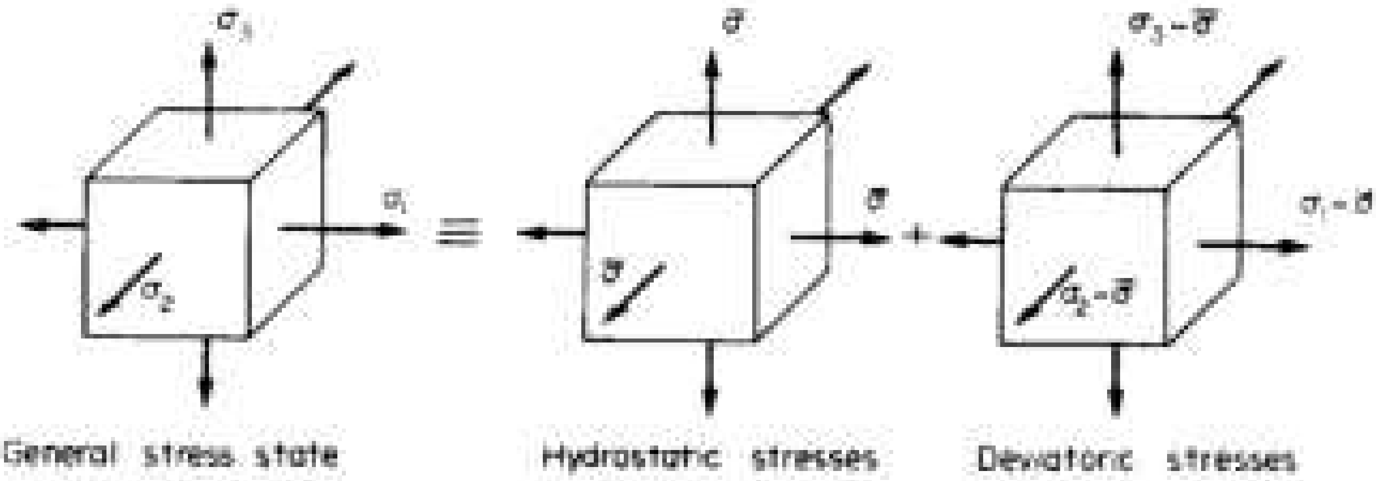


Fig. 14.27. Resolution of general three-dimensional principal stress state into "hydrostatic" and "deviatoric" components.

For convenience the principal stresses may be

written in terms of a mean stress $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ and additional shear stress terms,

i.e.

$$\sigma_1 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(\sigma_1 - \sigma_2) + \tfrac{1}{3}(\sigma_1 - \sigma_3)$$
$$\sigma_2 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(\sigma_2 - \sigma_1) + \tfrac{1}{3}(\sigma_2 - \sigma_3)$$
$$\sigma_3 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(\sigma_3 - \sigma_1) + \tfrac{1}{3}(\sigma_3 - \sigma_2)$$

The mean stress term may be considered as a *hydrostatic* tensile stress, equal in all directions, the strains associated with this giving rise to no distortion, i.e. the unit cube under the action of the hydrostatic stress alone would be strained into a cube. The hydrostatic stresses are sometimes referred to as the *spherical* or *dilatational* stresses.

The strain energy associated with the hydrostatic stress is termed the *volumetric strain energy* and is found by substituting

$$\sigma_1 = \sigma_2 = \sigma_3 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

into eqn. (14.21),

i.e. $$\text{volumetric strain energy} = \frac{3}{2E}\left[\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2\right](1 - 2\nu)$$

$$\therefore \quad U_v = \frac{(1 - 2\nu)}{6E}\left[(\sigma_1 + \sigma_2 + \sigma_3)^2\right] \text{ per unit volume}$$

Also known as dilatational strain energy

The remaining terms in the modified principal stress equations are shear stress terms (i.e. functions of principal stress differences in the various planes) and these are the only stresses which give rise to distortion of the stressed element.

They are therefore termed ***distortional or deviatoric stresses.***

Total strain energy per unit volume = shear strain energy per unit volume + volumetric strain energy per unit volume
i.e.

$$U_t = U_s + U_v$$

Therefore shear strain energy per unit volume is given by:

$$U_s = U_t - U_v$$

$$U_s = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] - \frac{(1-2\nu)}{6E}[(\sigma_1 + \sigma_2 + \sigma_3)^2]$$

This simplifies to

$$U_s = \frac{(1+\nu)}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

and, since $E = 2G(1+\nu)$,

$$U_s = \frac{1}{12G}[\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

or, alternatively,

$$U_s = \frac{1}{6G}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

Also known as shear or distortional strain energy

SUDDENLY APPLIED LOADS

If a load P is applied gradually to a bar to produce an extension ***δ*** the load-extension graph **, the work done being given by**

U=½ Pδ

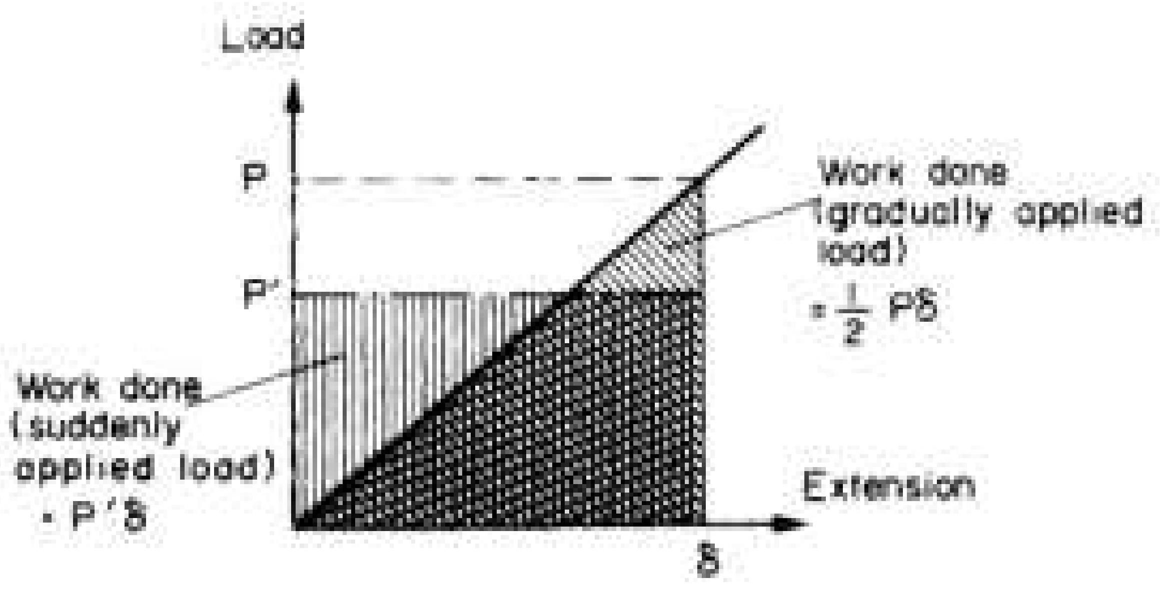


Fig. 11.6. Work done by a suddenly applied load.

- The bar will be strained by an equal amount δ in both cases and the energy stored must therefore be equal

$$P'\delta = \frac{1}{2}P\delta$$

$$P' = \frac{1}{2}P$$

It is then clear that vice versa ***a load P which is suddenly applied will produce twice the effect of the same load statically applied.***

Great care must be exercised, therefore, in the design of, for example, machine parts to exclude the possibility of sudden applications of load since associated stress levels are likely to be doubled.

Impact loads - axial load application

- When the load is dropped it will produce a maximum instantaneous extension $\boldsymbol{\delta}$ of the bar, and will therefore have done work (neglecting the mass of the bar and collar) = force **x distance = W (h + $\boldsymbol{\delta}$)**

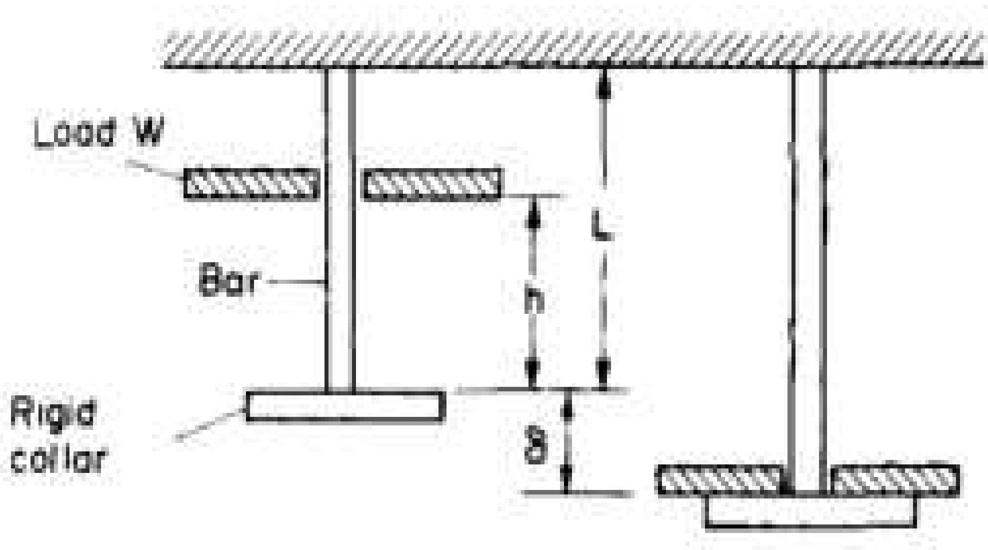


Fig. 11.7. Impact load – axial application.

This work will be stored as strain energy and is given by eqn. (11.2):

$$U = \frac{\sigma^2 AL}{2E}$$

where σ is the instantaneous stress set up.

$$\therefore \qquad \frac{\sigma^2 AL}{2E} = W(h + \delta)$$

If the extension δ is small compared with h it may be ignored and then, approximately,

$$\sigma^2 = 2WEh/AL$$

i.e.

$$\sigma = \sqrt{\left(\frac{2WEh}{AL}\right)} \tag{11.10}$$

If, however, δ is not small compared with h it must be expressed in terms of σ, thus

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma L}{\delta} \quad \text{and} \quad \delta = \frac{\sigma L}{E}$$

Therefore substituting in eqn. (11.9)

$$\frac{\sigma^2 AL}{2E} = Wh + \frac{W\sigma L}{E}$$

$$\therefore \qquad \frac{\sigma^2 AL}{2E} - \sigma\frac{WL}{E} - Wh = 0$$

$$\sigma^2 - \frac{2W}{A}\sigma - \frac{2WEh}{AL} = 0$$

Solving by "the quadratic formula" and ignoring the negative sign,

$$\sigma = \tfrac{1}{2}\left\{\frac{2W}{A} + \sqrt{\left[\left(\frac{2W}{A}\right)^2 + 4\left(\frac{2WEh}{AL}\right)\right]}\right.$$

i.e.
$$\sigma = \frac{W}{A} + \sqrt{\left[\left(\frac{W}{A}\right)^2 + \frac{2WEh}{AL}\right]}$$

This is the *accurate* equation for the *maximum* stress set up, and should always be used if there is any doubt regarding the relative magnitudes of δ and h.

Instantaneous extensions can then be found from

$$\delta = \frac{\sigma L}{E}$$

If the load is not dropped but *suddenly applied* from effectively zero height, $h = 0$, and eqn. (11.11) reduces to

$$\sigma = \frac{W}{A} + \frac{W}{A} = \frac{2W}{A}$$

CASTIGLIANO'S THEOREM FOR DEFLECTION

- If ***the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,***

i.e. if ***U is the total strain energy,***

the deflection in the direction of load W

$$= \partial U / \partial W.$$

Proof

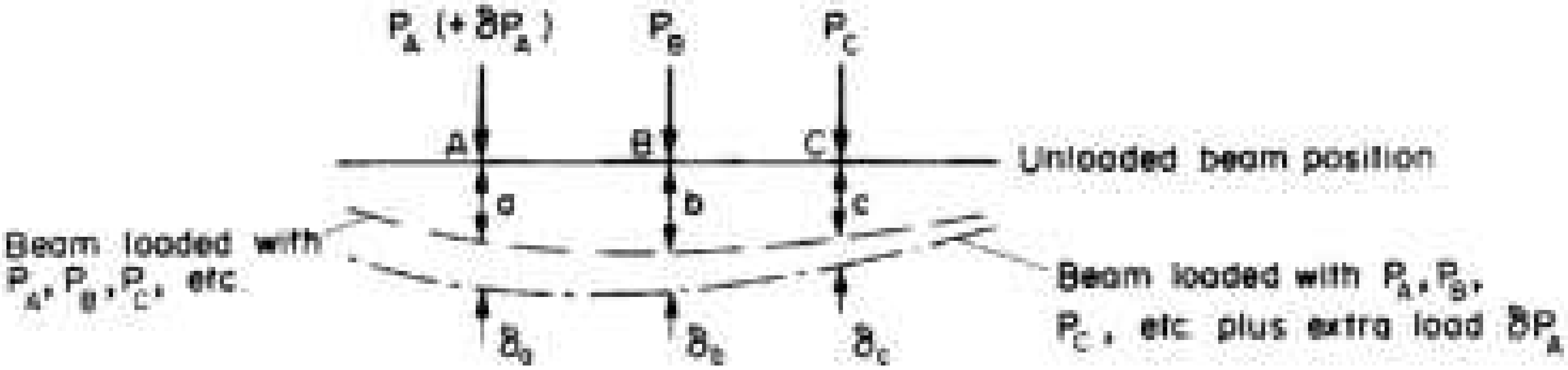


Fig. 11.9. Any beam or structure subjected to a system of applied concentrated loads $P_A, P_B, P_C \ldots P_N$, etc.

$$U = \tfrac{1}{2}P_A a + \tfrac{1}{2}P_B b + \tfrac{1}{2}P_C c + \ldots$$

If one of the loads, P_A, is now increased by an amount δP_A the changes in deflections will be δa, δb and δc, etc., as shown in Fig. 11.9.

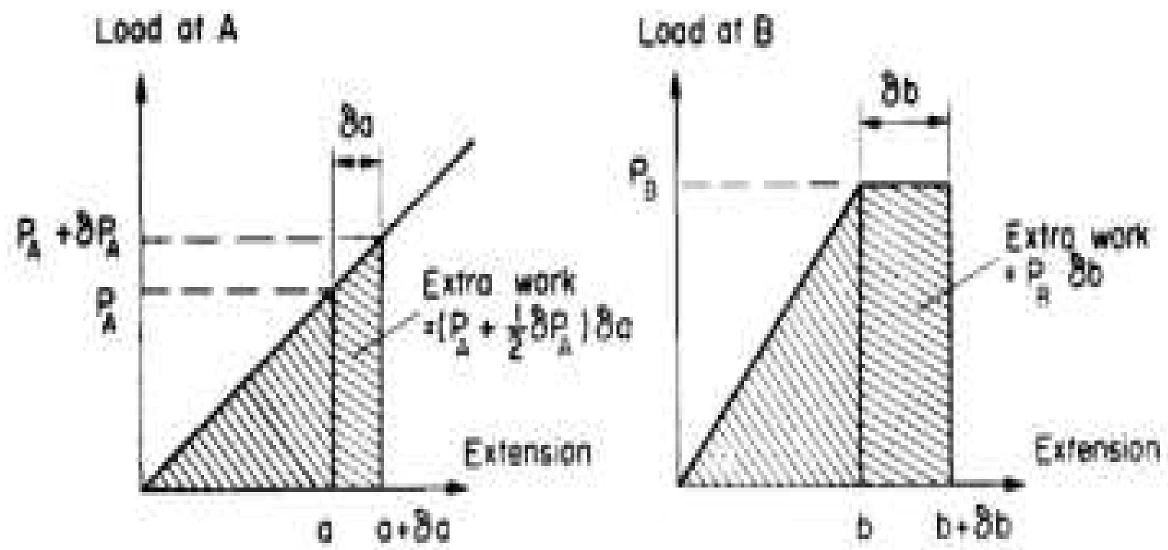


Fig. 11.10. Load–extension curves for positions A and B.

Extra work done at A (see Fig. 11.10)

$$= (P_A + \tfrac{1}{2}\delta P_A)\delta a$$

Extra work done at B, C, etc. (see Fig. 11.10)

$$= P_B\,\delta b,\ P_C\,\delta c, \text{ etc.}$$

Increase in strain energy

$$= \text{total extra work done}$$

$$\therefore \qquad \delta U = P_A\,\delta a + \tfrac{1}{2}\delta P_A\,\delta a + P_B\,\delta b + P_C\,\delta c + \ \ldots$$

and neglecting the product of small quantities

$$\delta U = P_A\,\delta a + P_B\,\delta b + P_C\,\delta c + \ \ldots$$

But if the loads $P_A + \delta P_A$, P_B, P_C, etc., were applied gradually from zero the total strain energy would be

$$U + \delta U = \sum \tfrac{1}{2} \times \text{load} \times \text{extension}$$

$$U + \delta U = \tfrac{1}{2}(P_A + \delta P_A)(a + \delta a) + \tfrac{1}{2}P_B(b + \delta b) + \tfrac{1}{2}P_C(c + \delta c) + \ \ldots$$

$$= \tfrac{1}{2}P_A\,a + \tfrac{1}{2}P_A\,\delta a + \tfrac{1}{2}\,\delta P_A\,a + \tfrac{1}{2}\delta P_A\,\delta a + \tfrac{1}{2}P_B\,b + \tfrac{1}{2}P_B\,\delta b + \tfrac{1}{2}P_C\,c + \tfrac{1}{2}P_C\,\delta c + \ \ldots$$

Neglecting the square of small quantities $(\tfrac{1}{2}\delta P_A \delta a)$ and subtracting eqn. (11.14),

$$\delta U = \tfrac{1}{2}\delta P_A\,a + \tfrac{1}{2}P_A\,\delta a + \tfrac{1}{2}P_B\,\delta b + \tfrac{1}{2}P_C\,\delta c + \ \ldots$$

or

$$2\delta U = \delta P_A\,a + P_A\,\delta a + P_B\,\delta b + P_C\,\delta c + \ \ldots$$

Subtracting eqn. (11.15),

$$\delta U = \delta P_A a \quad \therefore \quad \frac{\delta U}{\delta P_A} = a$$

or, in the limit,

$$\frac{\partial U}{\partial P_A} = a$$

i.e. the partial differential of the strain energy U with respect to P_A gives the deflection under and in the direction of P_A. Similarly,

$$\frac{\partial U}{\partial P_B} = b \quad \text{and} \quad \frac{\partial U}{\partial P_C} = c, \text{ etc.}$$

In most beam applications the strain energy, and hence the deflection, resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

$$\therefore \qquad U = \int \frac{M^2}{2EI}\, ds$$

$$\frac{\partial U}{\partial P} = \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial P} = \int \frac{2M}{2EI}\, ds \times \frac{\partial M}{\partial P}$$

i.e.

$$\delta = \frac{\partial U}{\partial P} = \int \frac{M}{EI} \frac{\partial M}{\partial P}\, ds \qquad (11.16)$$

which is the usual form of Castigliano's first theorem. The integral is evaluated as it stands to give the deflection under an existing load P, the value of the bending moment M at some general section having been determined in terms of P. If no general expression for M in terms

In cases where the deflection is required at a point or in a direction in which there is no load applied, an imaginary load P is introduced in the required direction, the integral obtained in terms of P and then evaluated with P equal to zero.

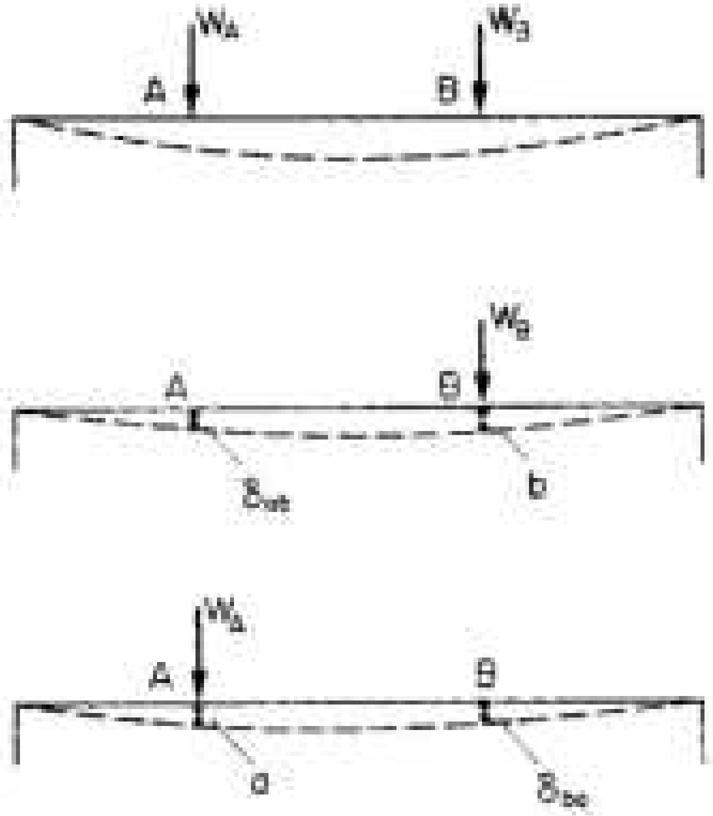
Application of Castigliano's theorem to angular movements

If the total strain energy, expressed in terms of the external moments, be partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction,

$$\theta = \int \frac{M}{EI} \frac{\partial M}{\partial M_i} ds \qquad (11.18)$$

where M_i is the imaginary or applied moment at the point where θ is required.

Maxwell's theorem of reciprocal displacements



δ_{ab} = deflection at A with load at B
δ_{ba} = deflection at B with load at A

. 5.24. Maxwell's theorem of reciprocal displacements.

Consider a beam subjected to two loads W_A and W_B at points A and B respectively as shown in Fig. 5.24. Let W_A be gradually applied first, producing a deflection a at A.

$$\text{Work done} = \tfrac{1}{2} W_A a$$

When W_B is applied it will produce a deflection b at B and an additional deflection δ_{ab} at A (the latter occurring in the presence of a now constant load W_A).

$$\text{Extra work done} = \tfrac{1}{2} W_B b + W_A \delta_{ab}$$

$$\therefore \quad \text{total work done} = \tfrac{1}{2} W_A a + \tfrac{1}{2} W_B b + W_A \delta_{ab}$$

Similarly, if the loads were applied in reverse order and the load W_A at A produced an additional deflection δ_{ba} at B, then

$$\text{total work done} = \tfrac{1}{2}W_B b + \tfrac{1}{2}W_A a + W_B\delta_{ba}$$

It should be clear that, regardless of the order in which the loads are applied, the total work done must be the same. Inspection of the above equations thus shows that

$$W_A\delta_{ab} = W_B\delta_{ba}$$

If the two loads are now made equal, then

$$\delta_{ab} = \delta_{ba} \tag{5.22}$$

i.e. *the deflection at A produced by a load at B equals the deflection at B produced by the same load at A*. This is Maxwell's theorem of reciprocal displacements.

a typical example of the application of this theorem

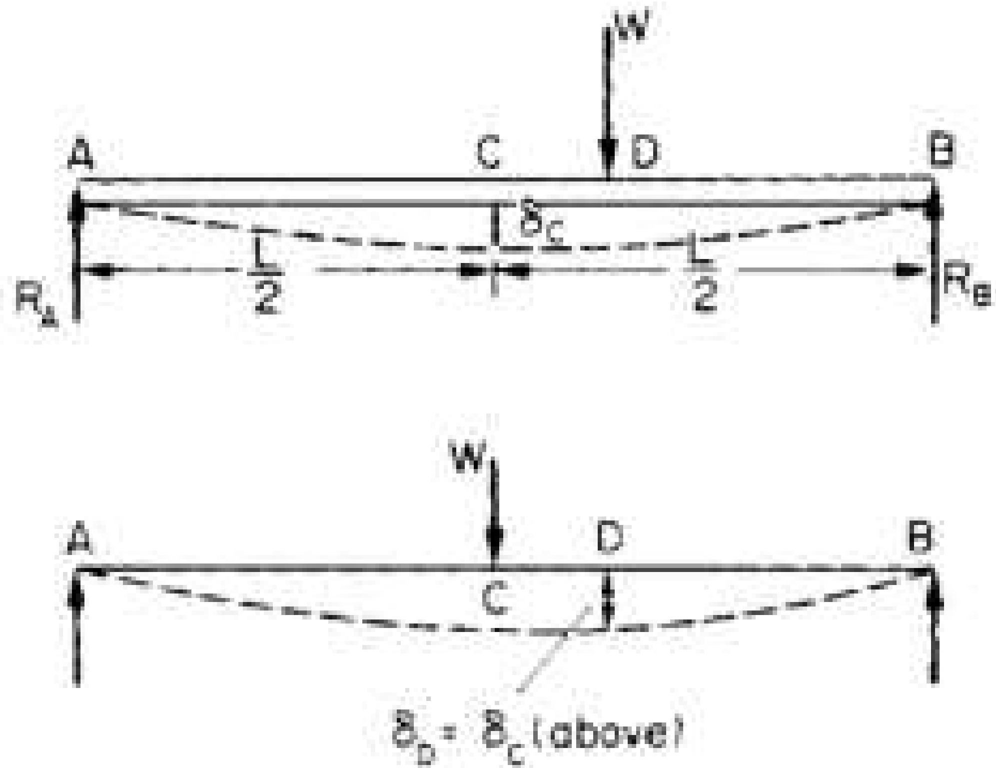


Fig. 5.25.

Maxwell’s theorem of reciprocal displacements can also be applied if one or both of the loads are replaced by moments or couples. In this case it can be shown that the theorem is modified to the relevant one of the following forms **(a), (b):**

(a) ***The angle of rotation at A due to a concentrated force at B is numerically equal to the deflection at B due to a couple at A provided that the force and couple are also numerically equal (Fig. 5.26).***

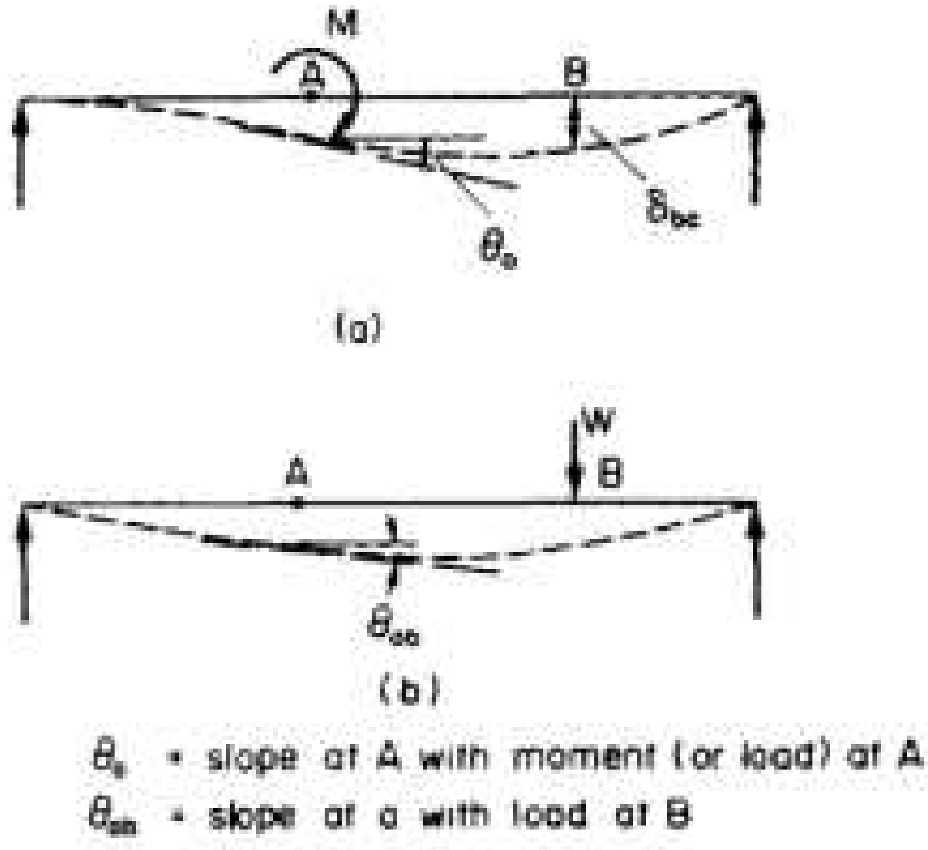


Fig. 5.26.

(b) ***The angle of rotation at A due to a couple at B is equal to the rotation at B due to the same couple applied at A (Fig. 5.27).***

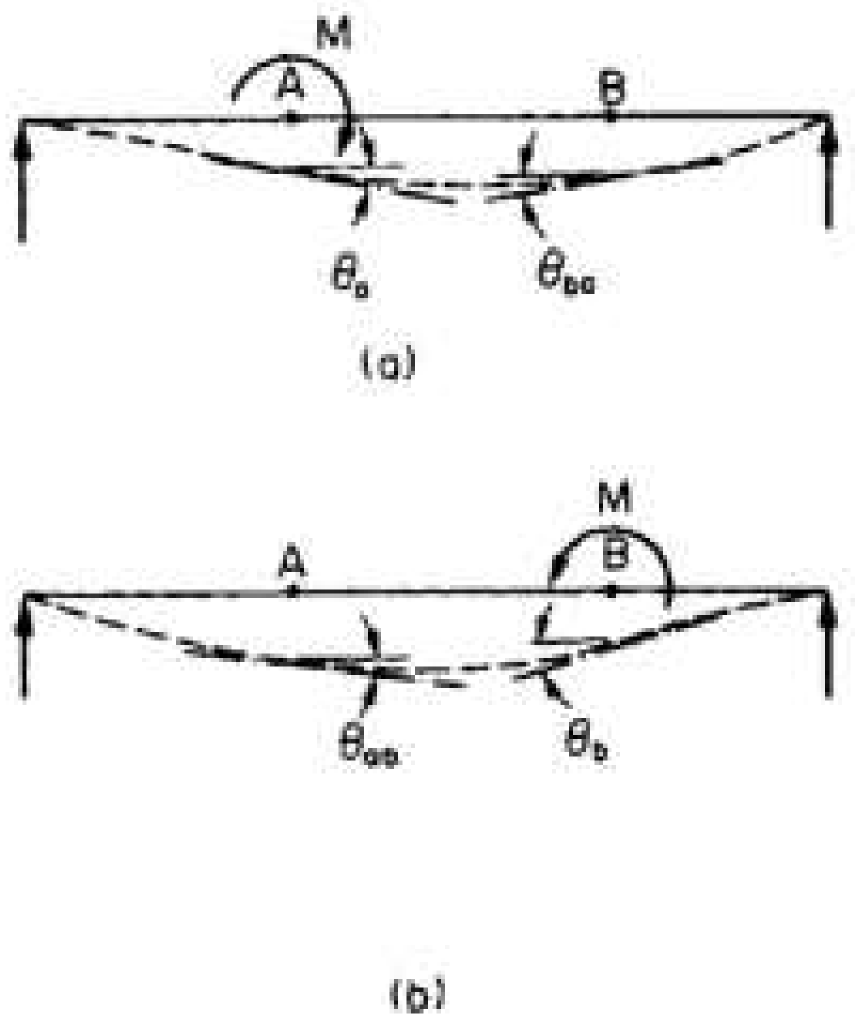


Fig. 5.27.

Thin cylinders

SOM –I

THIN-WALLED PRESSURE VESSELS

Cylindrical and spherical pressure vessels are commonly used for storing gas and liquids under pressure.

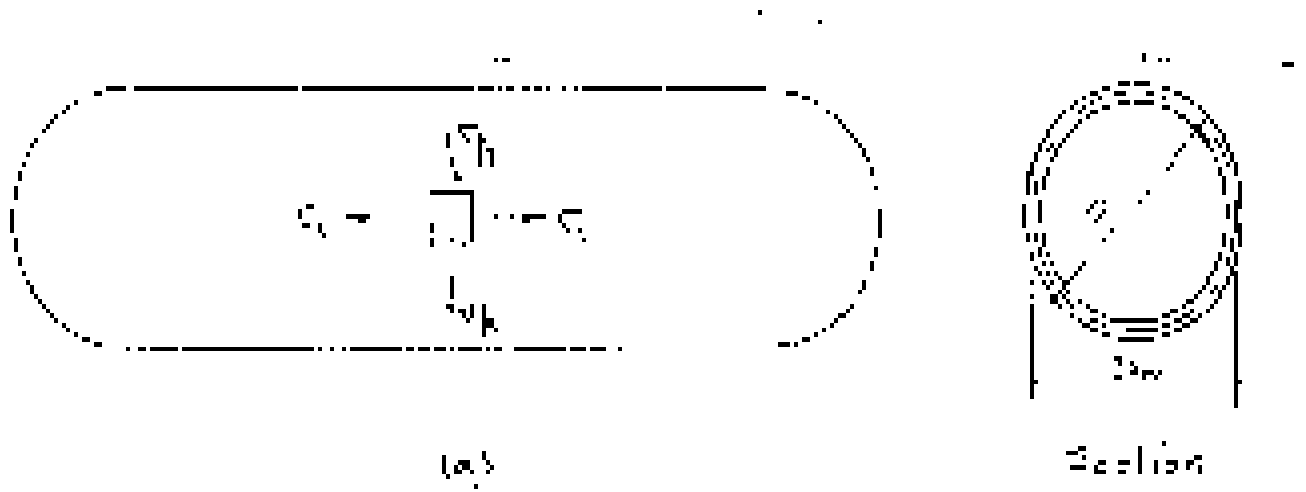
A thin cylinder is normally defined as one in which the thickness of the metal is less than 1/20 of the diameter of the cylinder.

THIN-WALLED PRESSURE VESSELS CONTD

In thin cylinders, it can be assumed that the variation of stress within the metal is negligible, and that the mean diameter, D_m is approximately equal to the internal diameter, D.

At mid-length, the walls are subjected to hoop or circumferential stress, and a longitudinal stress.

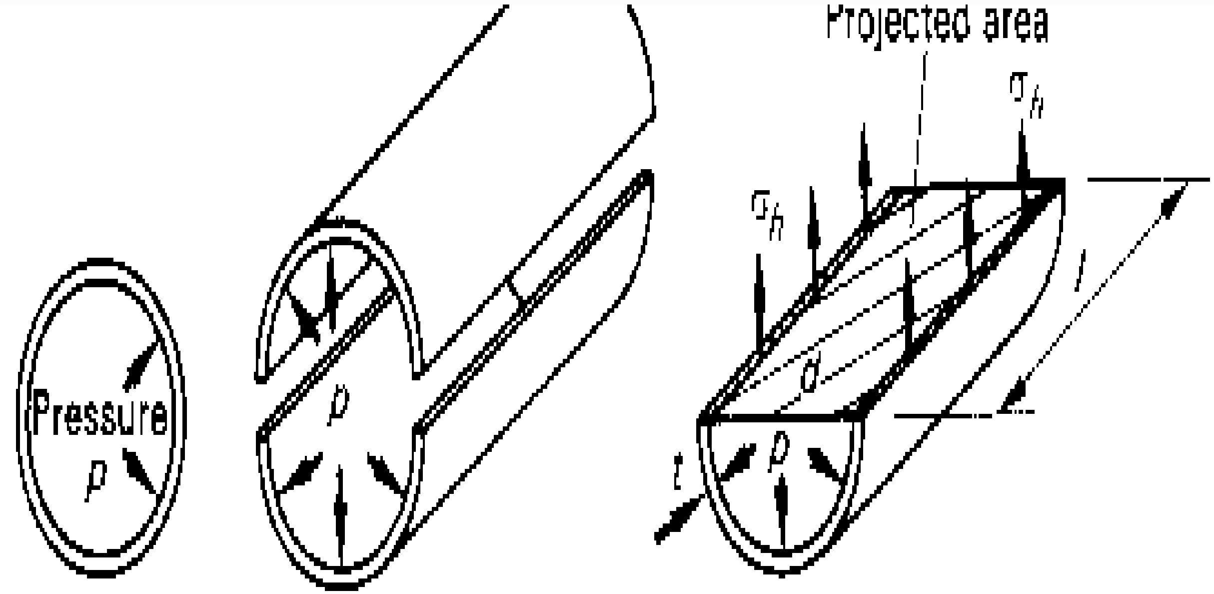
Hoop and Longitudinal Stress



(a)

Section

Hoop stress in thin cylindrical shell



Radial stress equal to pressure

(a)

Failure along longitudinal section

(b)

Hoop stress in thin cylindrical shell Contd.

The internal pressure, p tends to increase the diameter of the cylinder and this produces a hoop or circumferential stress (tensile).

If the stress becomes excessive, failure in the form of a longitudinal burst would occur.

Hoop stress in thin cylindrical shell Concluded

Consider the half cylinder shown. Force due to internal pressure, p is balanced by the force due to hoop stress, σ_h .

i.e. hoop stress x area = pressure x projected area

σ_h x 2 L t = P x d L

$$\boxed{\sigma_h = (P\,d) / 2\,t}$$

Where: d is the internal diameter of cylinder; t is the thickness of wall of cylinder.

2.2.2. Longitudinal stress in thin cylindrical shell

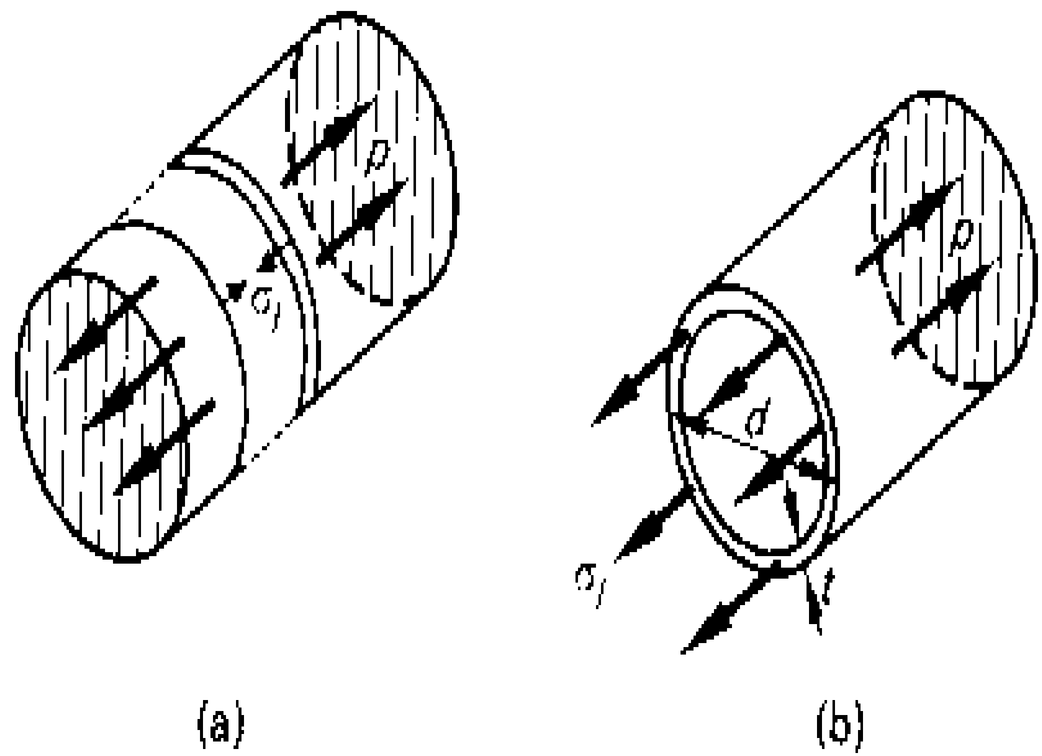


Fig. 3.15 Longitudinal stress in a thin cylindrical shell

Longitudinal stress in thin cylindrical shell Contd.

The internal pressure, P also produces a tensile stress in longitudinal direction as shown above.

Force by P acting on an area $\frac{\pi d^2}{4}$ is balanced by

longitudinal stress, σ_L acting over an approximate area,

$\pi d t$ (mean diameter should strictly be used). That is:

$$\sigma_L \ x \ \pi d t = P \ x \ \frac{\pi d^2}{4}$$

$$\sigma_L = \frac{P d}{4 t}$$

Note

1. Since hoop stress is twice longitudinal stress, the cylinder would fail by tearing along a line parallel to the axis, rather than on a section perpendicular to the axis.

The equation for hoop stress is therefore used to determine the cylinder thickness.

Allowance is made for this by dividing the thickness obtained in hoop stress equation by efficiency (i.e. tearing and shearing efficiency) of the joint.

Longitudinal stress in thin cylindrical shell Concluded

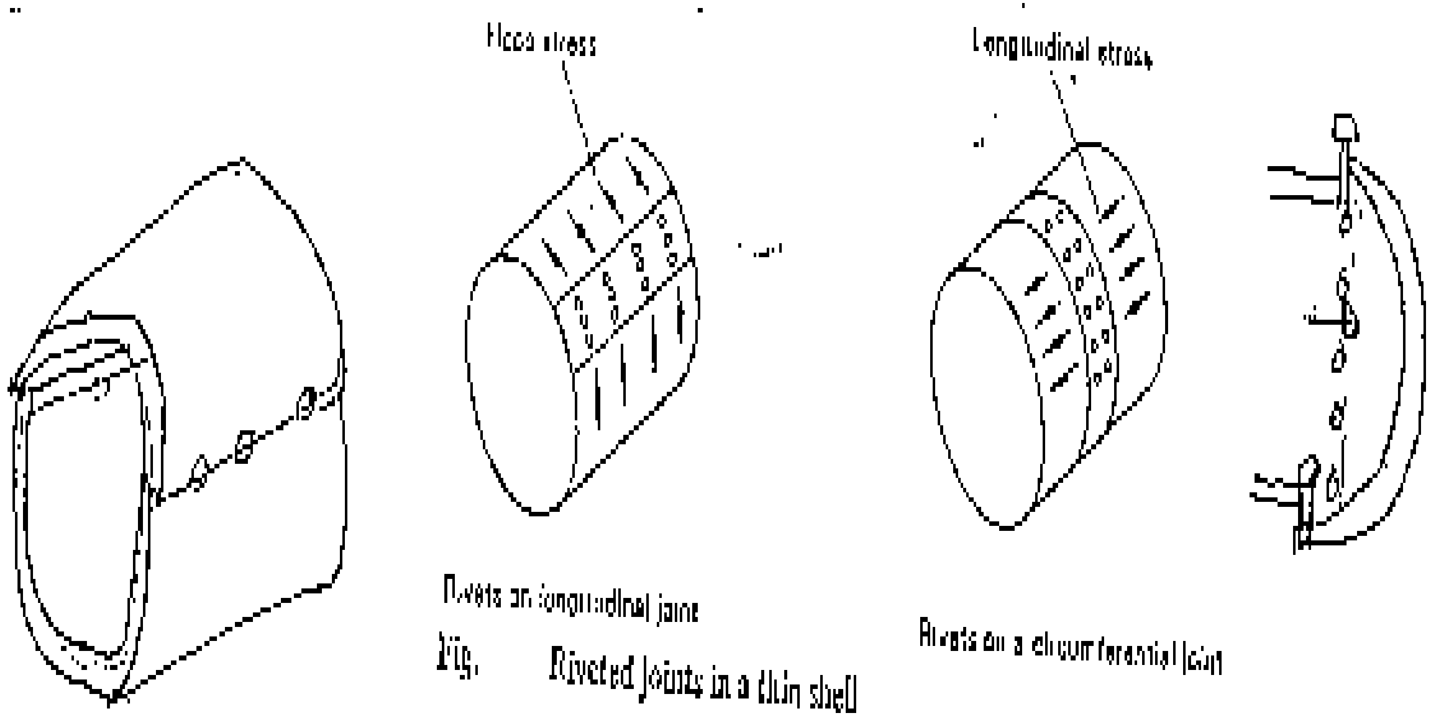


Fig. Riveted joints in a thin shell

Example

A cylindrical boiler is subjected to an internal pressure, p. If the boiler has a mean radius, r and a wall thickness, t, derive expressions for the hoop and longitudinal stresses in its wall. If Poisson’s ratio for the material is 0.30, find the ratio of the hoop strain to the longitudinal strain and compare it with the ratio of stresses.

Solution

Hoop stress will cause expansion on the lateral direction and is equal to σ_y while the longitudinal stress is σ_x

Hoop stress, $\sigma_h \quad = \dfrac{p\,d}{2\,t} = \dfrac{p\ x\ 2r}{2t} = \dfrac{p\ r}{t} \quad ie\sigma_y$

Longitudinal stress, $\quad \sigma_L = \dfrac{p\,d}{4t} = \dfrac{p\ x\ 2r}{4t} = \dfrac{p\ r}{2t} i.e.\sigma_x$

(a) Stress ratio = 2

(b) $\varepsilon_x = \dfrac{1}{E}[\sigma_x - \upsilon\,\sigma_y] = \dfrac{1}{E}[\dfrac{pr}{2t} - 0.3\,\dfrac{pr}{t}] = \dfrac{0.2}{E}\,\dfrac{pr}{t}(Longitudinal\ strain)$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \upsilon\,\sigma_x] = \frac{1}{E}[\frac{pr}{t} - 0.3\,\frac{pr}{2t}] = \frac{0.85}{E}\,\frac{pr}{t}(Hoop\ strain)$$

$$Ratio\ of\ \ strains = \frac{Hoop\ strain}{Longitudinal\ strain} = \frac{0.85}{0.2} = 4.25$$

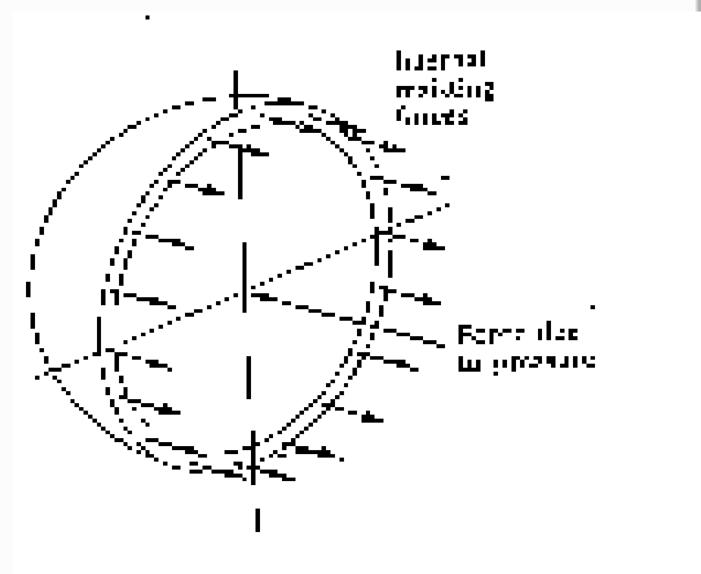
Pressure in Spherical Vessels

2.2.3 Pressure in Spherical Vessels

Problems dealing with spherical vessels follow similar solutions to that for thin cylinders except that there will be longitudinal stresses in all directions. No hoop or circumferential stresses are produced.

i.e

$$\boxed{\sigma_L = \frac{Pd}{4t}}$$



Volume Changes

Example: A pressure cylinder, 0.8 m long is made out of 5 mm thick steel plate which has an elastic modulus of 210×10^3 N/mm^2 and a Poisson's ratio of 0.28. The cylinder has a mean diameter of 0.3 m and is closed at its ends by flat plates. If it is subjected to an internal pressure of 3 N/mm^2, calculate its increase in volume.

SOLUTION

Hoop stress, σ_h = (P d) / 2 t =

$$\frac{3N/mm^2\ x\ 300\,mm}{2\,x\,5\,mm} = 90\,N/mm^2$$

Longitudinal stress, σ_L = (P d) / 4 t = 45 N/mm2

Longitudinal strain,

$$\varepsilon_L = \frac{1}{E}[\sigma_L - \upsilon\,\sigma_h] = \frac{1}{210\,x\,10^3\,N/mm^2}[45 - 0.28\,x\,90] = 0.00009429$$

SOLUTION CONCLUDED

Hoop strain,

$$\varepsilon_h = \frac{1}{E}[\sigma_h - \upsilon\,\sigma_L] = \frac{1}{210\,x\,10^3\,N/mm^2}[90 - 0.28\,x\,45] = 0.0003686$$

Volumetric strain = $\varepsilon_L + 2\,\varepsilon_h = 0.00083134$ (See Section 1.4)

Original volume of cylinder is equal to :

$$\frac{\pi\,x\,300^2}{4}\,x\,800 = 56.5487\,x\,10^{-6}\,mm^3$$

$$Increase\ in\ volume = 56.5487\,x\,10^{-6}\,x\,0.00083134 = 47009\,mm^3$$

Example

The dimensions of an oil storage tank with hemispherical ends are shown in the Figure. The tank is filled with oil and the volume of oil increases by 0.1% for each degree rise in temperature of 1^0C. If the coefficient of linear expansion of the tank material is 12×10^{-6} per ^0C, how much oil will be lost if the temperature rises by 10^0C.

SOLUTION

For 10^0C rise in temperature:

Volumetric strain of oil = 0.001 x 10 = 0.01

Volumetric strain of tank = $3\ \alpha\ T$

= 3 x 12 x 10^{-6} x 10 = 0.00036

Difference in volumetric strain = 0.01 - 0.00036 = 0.00964

Volume of tank = π x 102 x 100 + 4/3 π x x 103 =

$10000\pi + 1333.33\pi$

$= 11333.33\pi\ m^3$

Volume of oil lost = strain difference x volume of tank = $0.00964 \times 11333.33\pi\ m^3$

= 343.2 m^3.