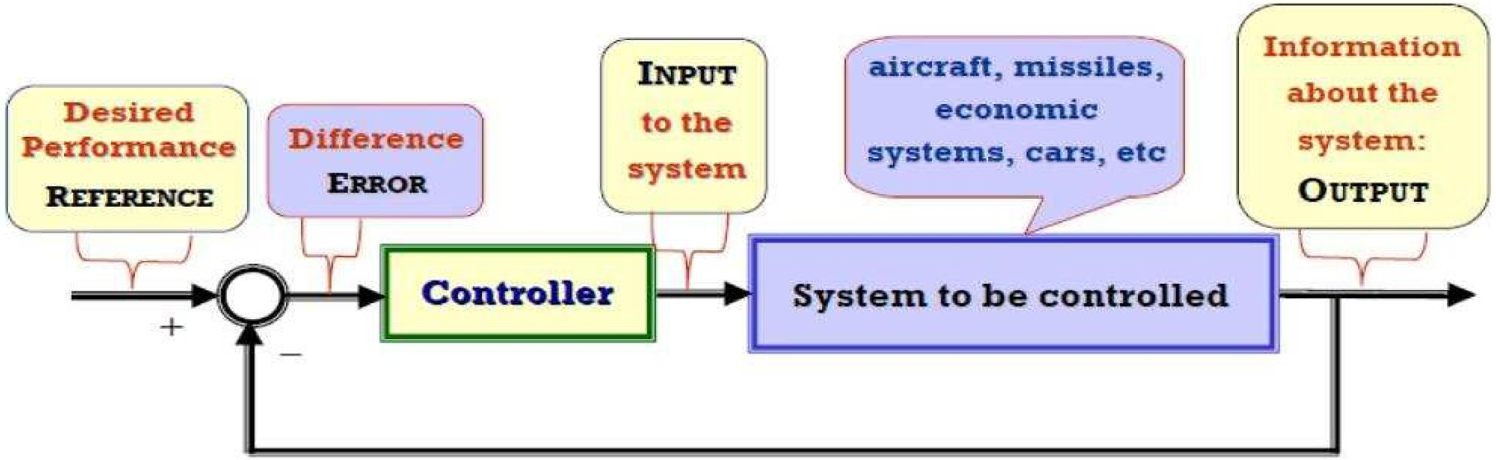
Sampled data Control Systems

Introduction

What is a control system?

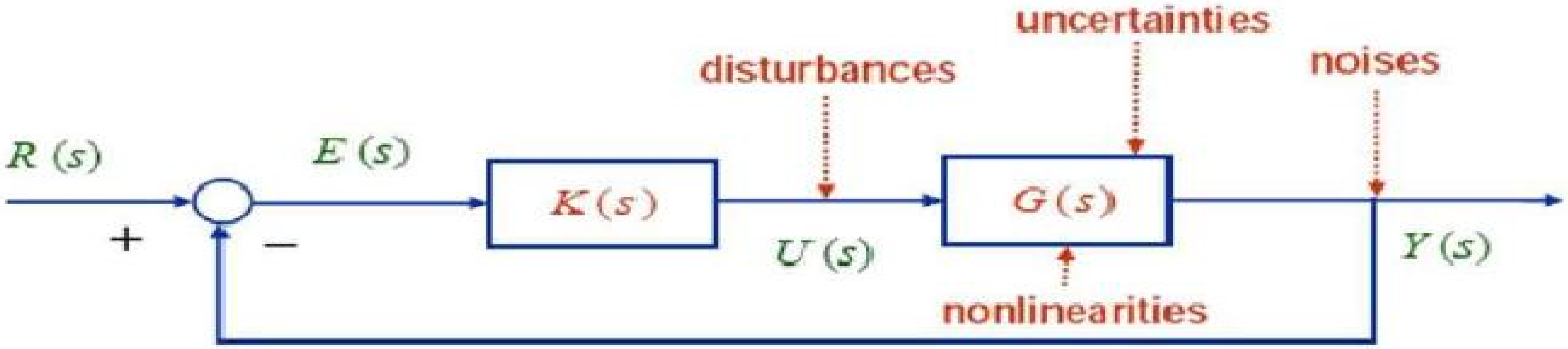


Objective:
To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

Key Issues:
1) How to describe the system to be controlled? (Modeling)
2) How to design the controller? (Control)

Introduction

What is important in a control system?



Stability

- **(Transient) response speed**
- **Accuracy**
  - dynamic overshooting and oscillation duration
  - Steady state error
- **Robustness**
  - errors in models (uncertainties and nonlinearities)
  - effects of disturbances
  - effects of noises

Introduction

Modeling of dynamic systems

Model: A representation of a system.

Types of Models:

- Physical models (prototypes)
- Mathematical models (e.g., input-output relationships)
    - Analytical models (using physical laws)
    - Computer (numerical) models
    - Experimental models (using input/output experimental data)

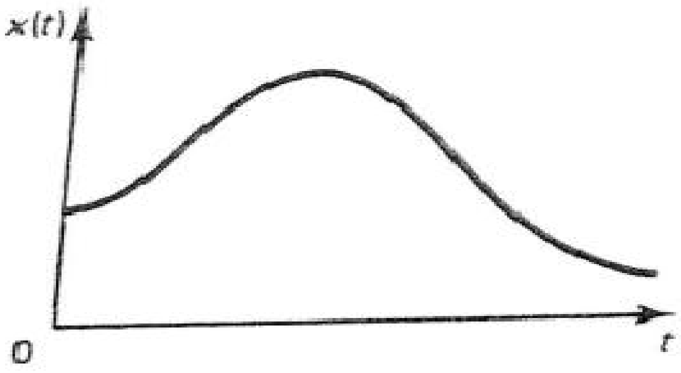
Models for physical dynamic systems:

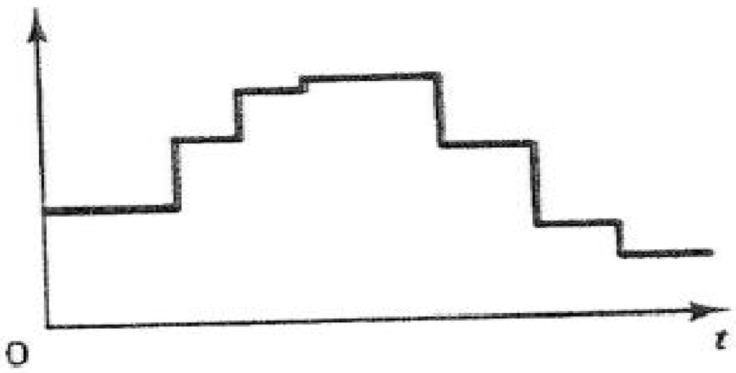
- Lumped-parameter models
- Continuous-parameter models. Example: Spring element (flexibility, inertia, damping)

Introduction

Signal categories for identifying control system types

Continuous-time signal & quantized signal





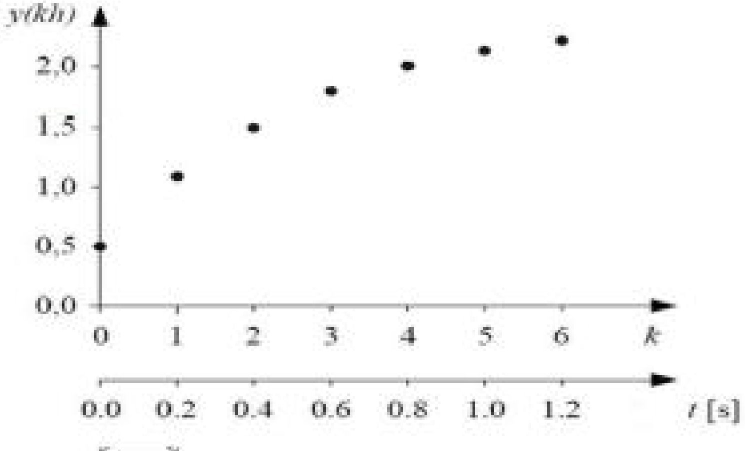
Continuous-time signal is defined continuously in the time domain. Figure on the left shows a continuous-time signal, represented by x(t).

Quantized signal is a signal whose amplitudes are discrete and limited. Figure on the right shows a quantized signal.

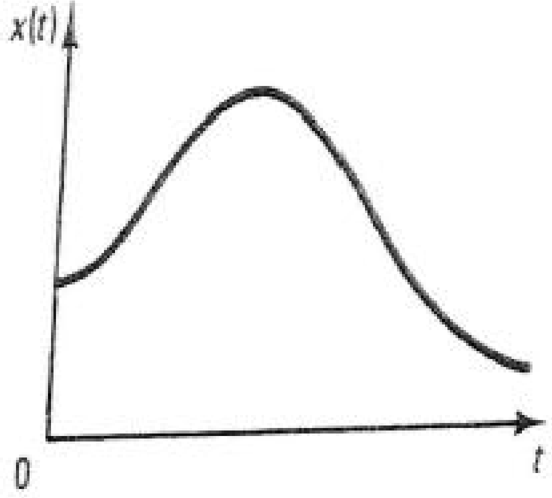
Analog signal or **continuous signal** is continuous in time and in amplitude. The real word consists of analog signals.

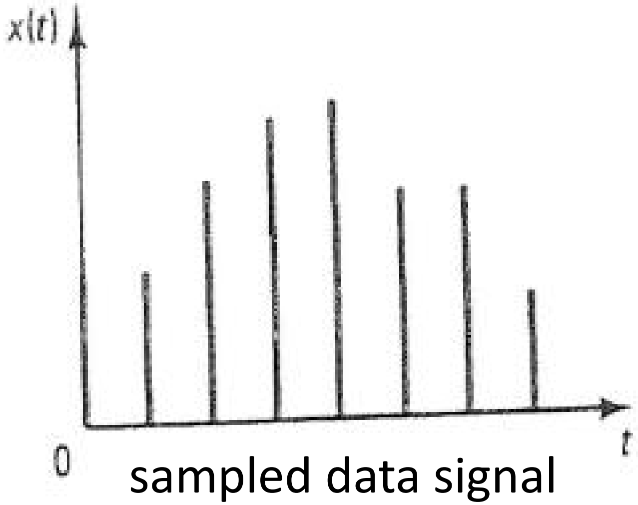
Introduction

Discrete-time signal & sampled-data signal



Discrete time signal





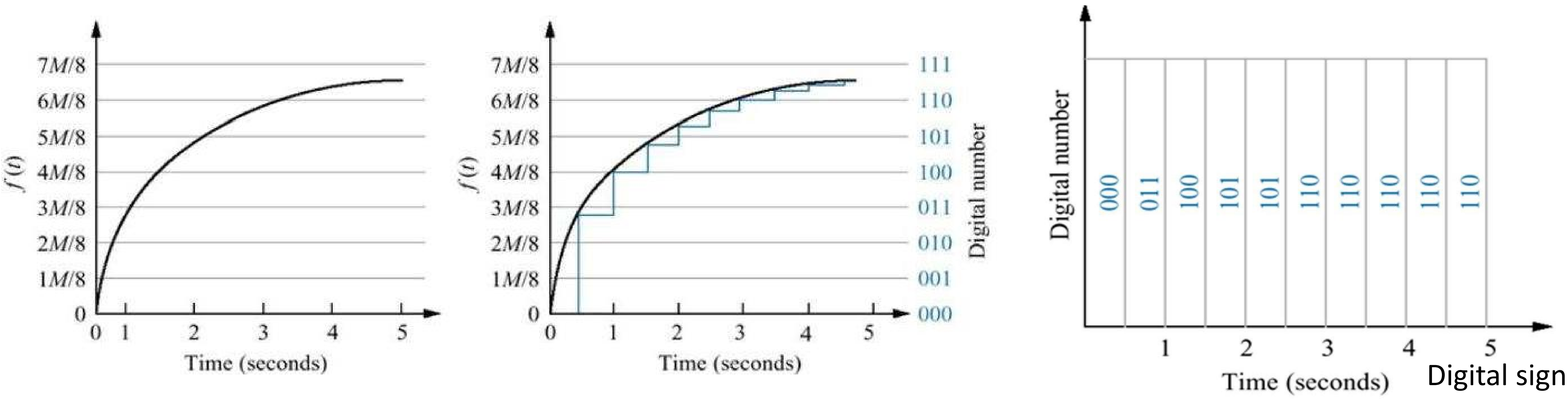
sampled data signal

Discrete-time signal is defined only at certain time instants. For a discrete-time signal, the amplitude between two consecutive time instants is just not defined. Figure on theleft shows a discrete-time signal, represented by $y(kh)$, or simply $y(k)$, where k is an integer and h is the time interval.

Sampled-data signal is a discrete-time signal resulting by sampling a continuous-time signal. Figure on the right shows a sampled-data signal deriving from the continuous-time signal, shown in the figure at the center, by a sampling process. It isrepresented by $x*(t)$.

Introduction

Digital signal or binary coded data signal



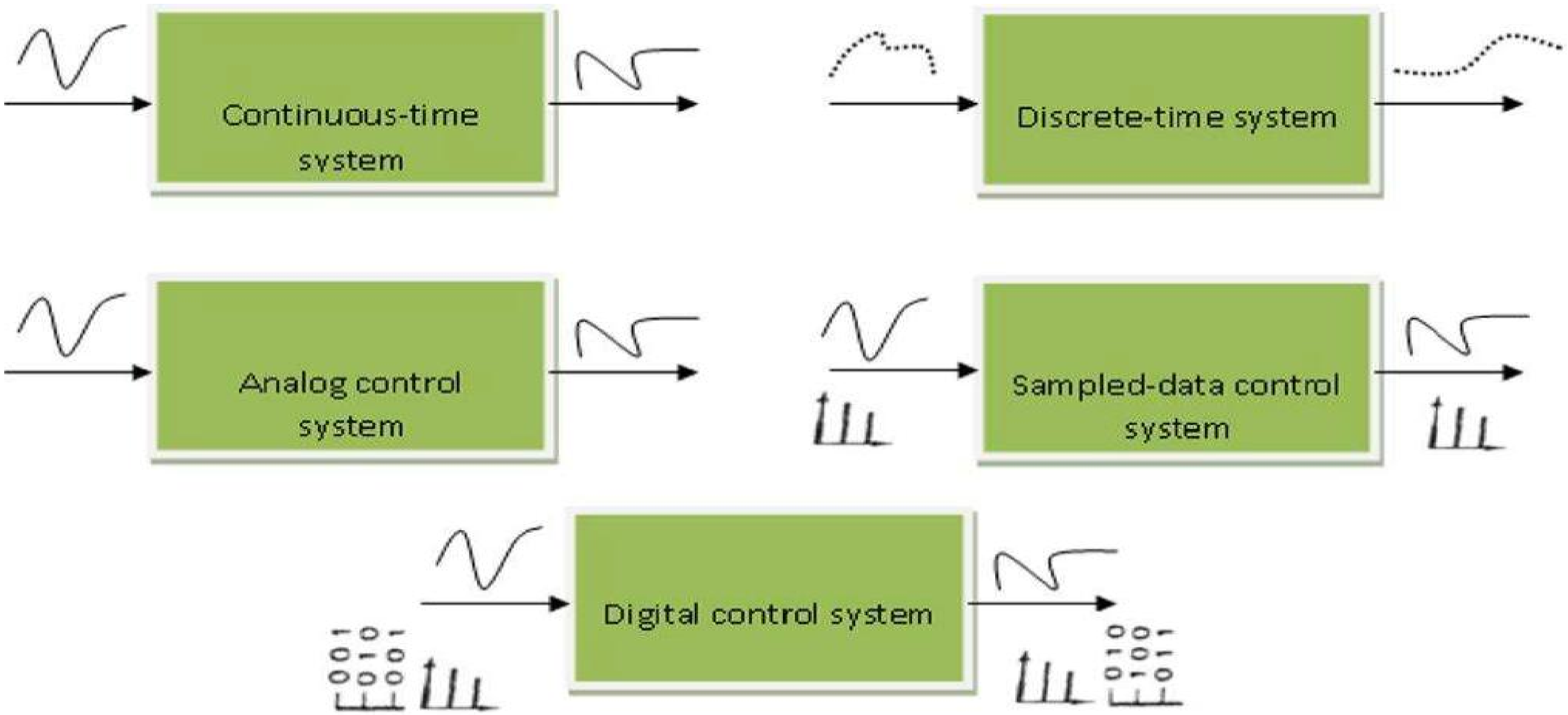
Digital signal

Digital signal is a sequence of binary numbers. In or out from a microprocessor, a semiconductor memory, or a shift register.

In practice, a digital signal, as shown in the figures at the bottom, is derived by two processes: **sampling and then quantizing.**

Introduction

Control System Types



Introduction

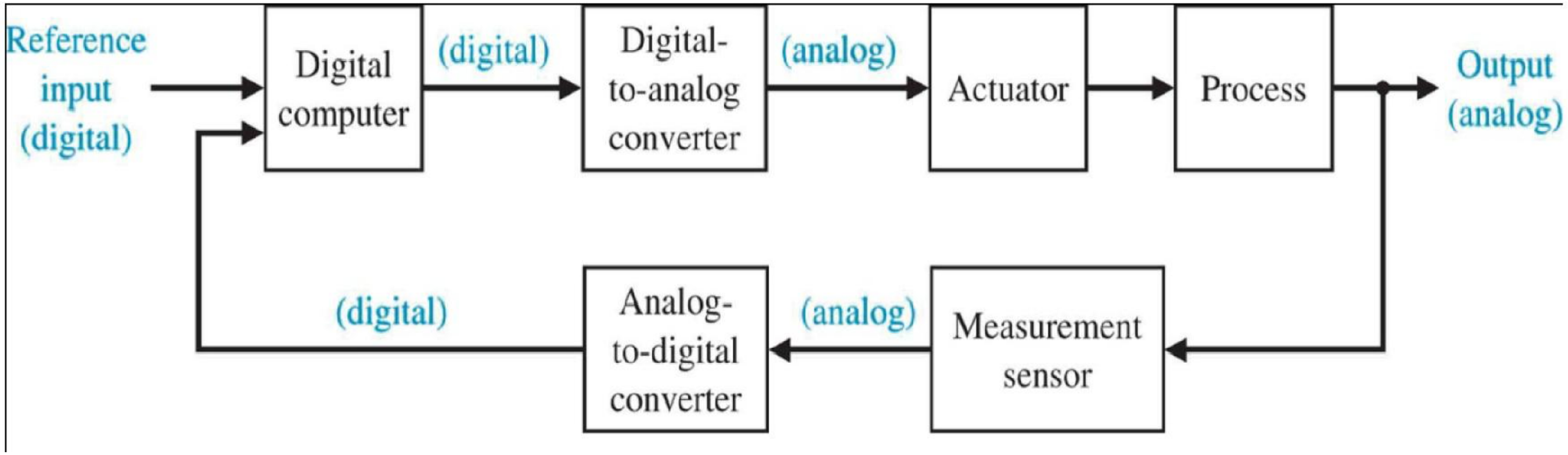
Mathematical comparison between analog and digital control systems

System	Analytical model		
	Time domain	Frequency domain	
Continuous-time systems	Differential equations or state-space equations	Laplace transfer function (s-transfer function)	s-plane analysis and design techniques (Routh-Hurwitz stability criterion, root locus techniques, Bode plots, etc.)
Discrete-time systems	Difference equations or discrete-time state-space equations	Impulse transform function (z-transfer function)	z-plane analysis and design techniques (Jury stability test, modified root locus techniques, etc.)

Introduction

Digital Control systems

Digital controls are used for achieving optimal performance-for example, in the form of maximum productivity, maximum profit, minimum cost, or minimum energy use.



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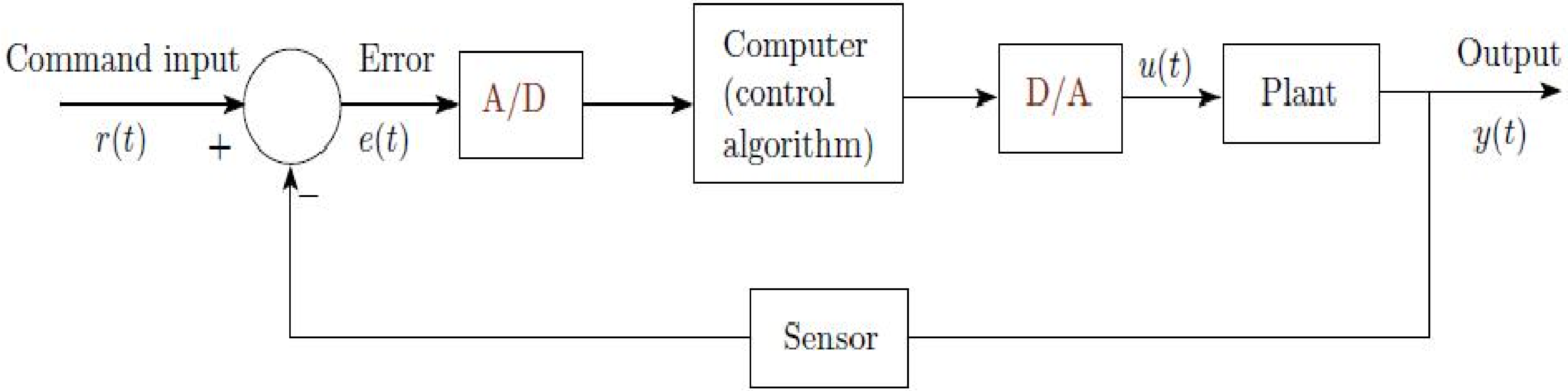


Introduction

Advantages of digital computers:

1. Reduced cost,
2. Flexibility in response to design changes,
3. Noise immunity
4. Digital control systems are more suitable for Modern control systems.

Disadvantages of digital computers:

1. From the tracking performance side, the analog control system exhibits good performances than digital control system.
2. Digital control system will introduce a delay in the loop.

Introduction

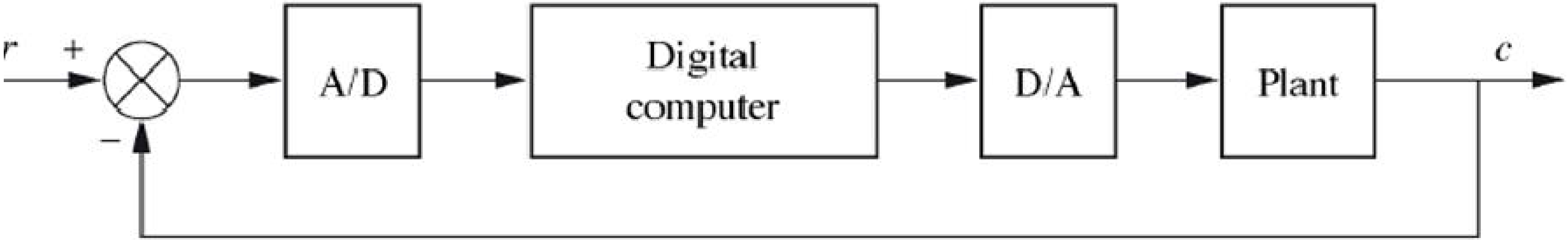
The loop (forward and feedback) contains both analog and digital signals must provide a means for conversion from one form to other to be used by each subsystem.

Analog-to-Digital Converter(ADC)

A device that converts analog signal to digital signal is called Analog-to-Digital Converter.

Digital Analog Converter(DAC)

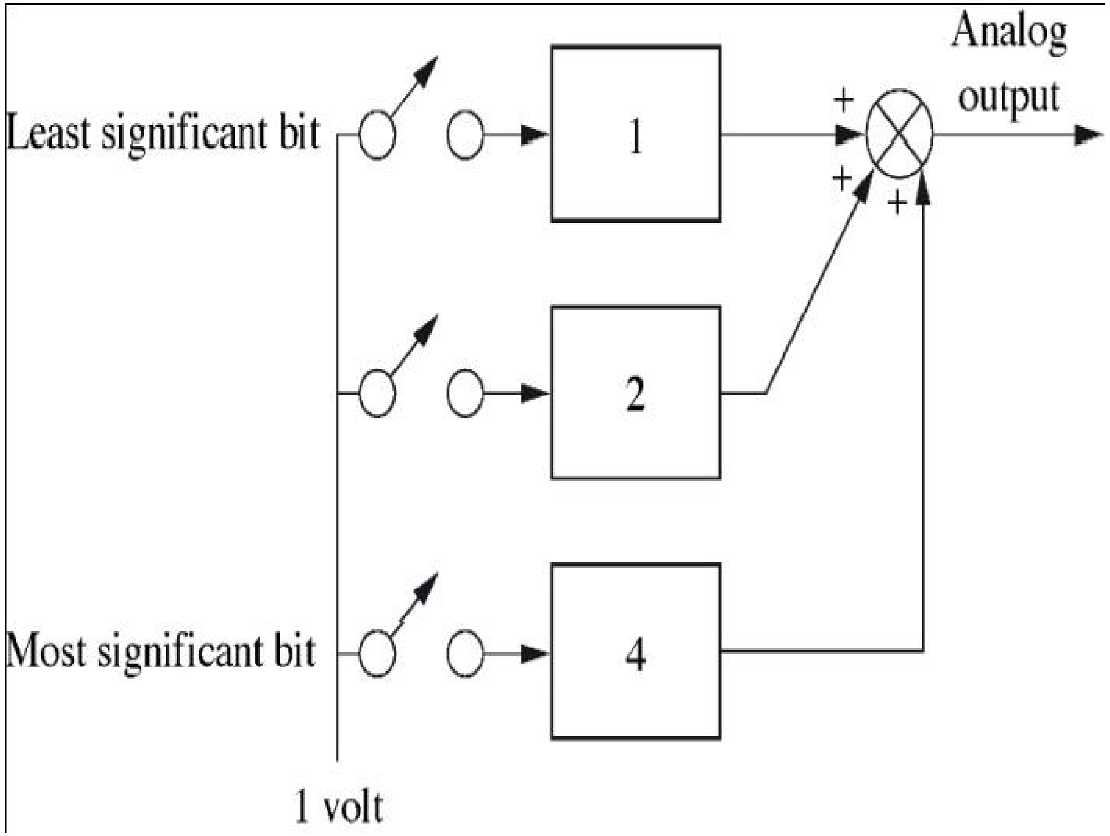
A device that converts digital signals to analog signals is called a Digital Analog Converter.



Introduction

Digital-To-Analog Conversion

From the binary number each bit is properly weighted voltages and are summed together to yield analogue output.



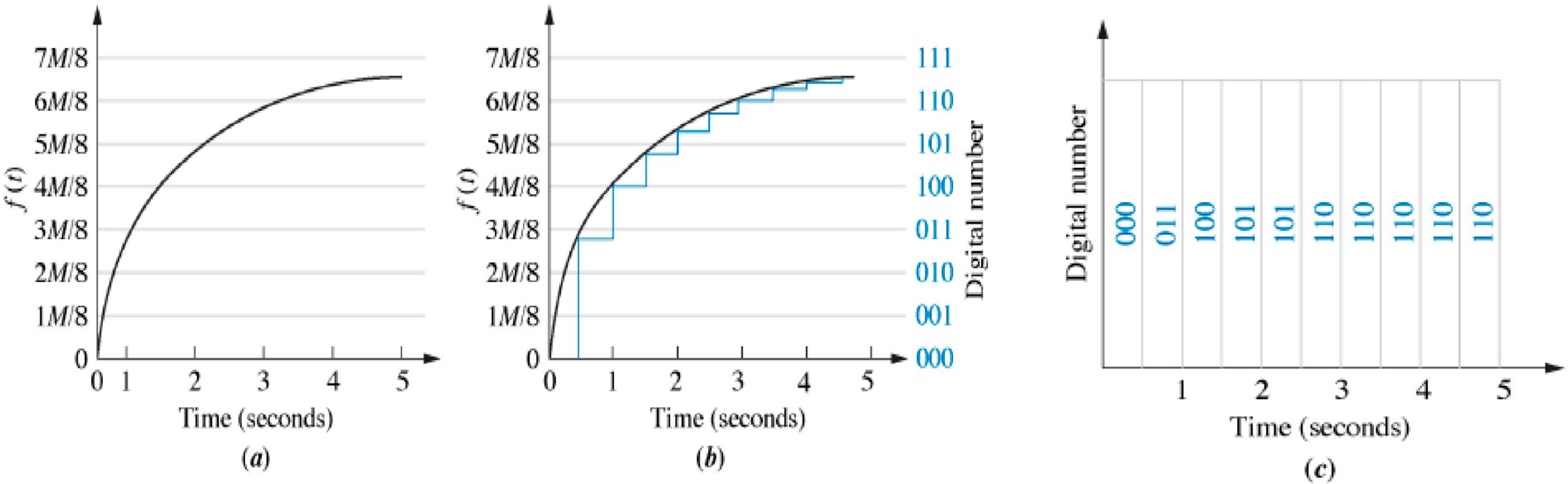
Introduction

Analog-To-Digital Conversion

ADC is not instantaneous and needs two-step process.

There is a delay between the analog input voltage and the output digital word.

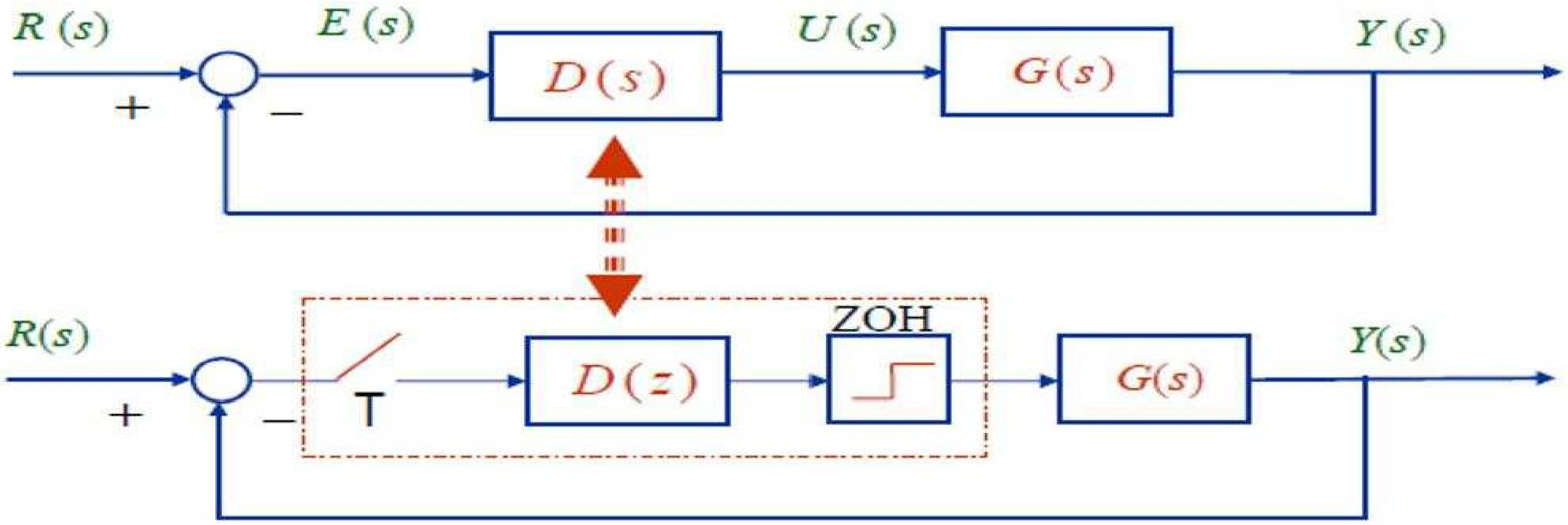
In ADC, the analog signal is first converted to a sampled signal and then converted to a sequence of binary numbers, the digital signal.



Introduction

Controller design in digital control systems - Design in S-domain

Digitization (DIG) or discrete control design

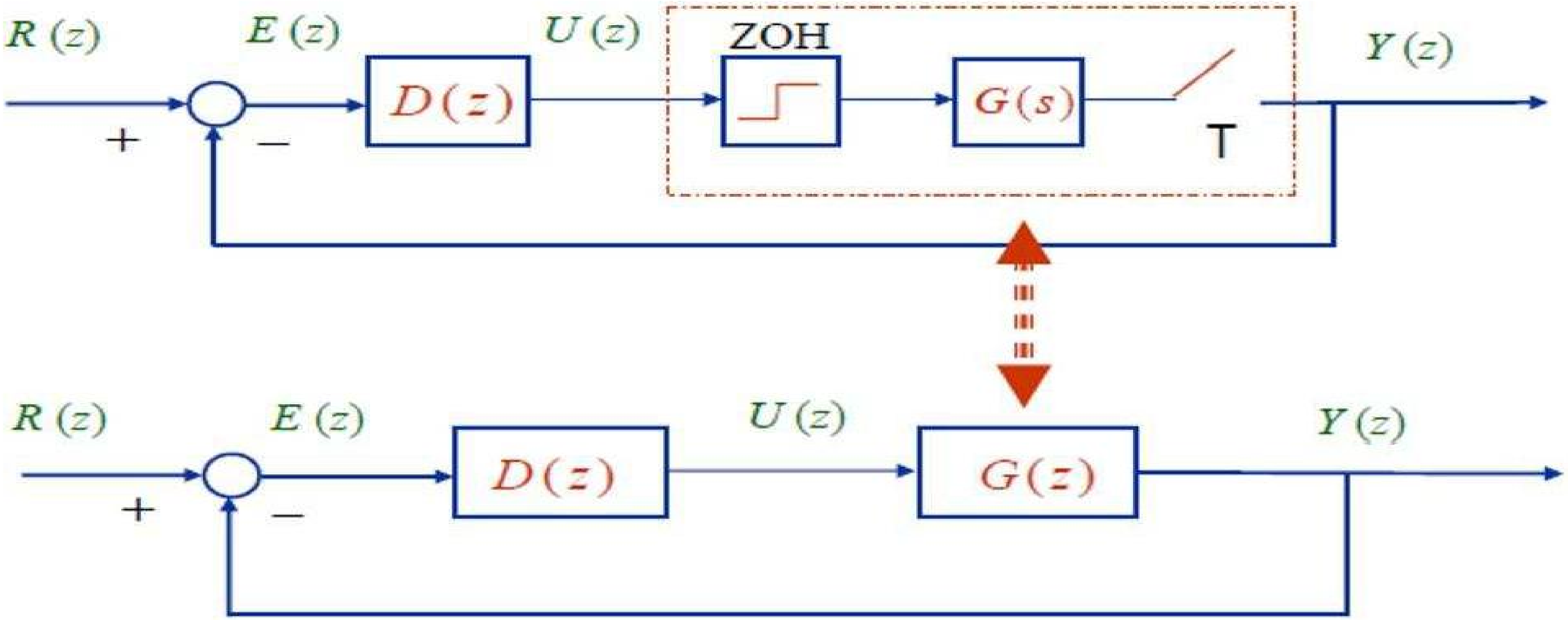


The above design works very well if sampling period T is sufficiently small.

Introduction

Controller design in digital control systems -Design in Z domain

Direct (DIR) control design



State Space Models

There are many alternative model formats that can be used for linear dynamic systems. In simple SISO problems, any representation is probably as good as any other. However, as we move to more complex problems (*especially multivariable problems*), it is desirable to use special model formats. One of the most flexible and useful structures is the state space model.

We will examine linear state space models in a little more depth for the SISO case. Many of the ideas will carry over to the MIMO case which we will study later. In particular we will study

- similarity transformations and equivalent state representations,
- state space model properties:
  - controllability, reachability, and stabilizability,
  - observability, reconstructability, and detectability,
- special (*canonical*) model formats.

Linear Continuous-Time State Space Models

A continuous-time linear time-invariant state space model takes the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \qquad\qquad x(t_o) = x_o$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control signal, $y \in \mathbb{R}^p$ is the output, $x_0 \in \mathbb{R}^n$ is the state vector at time $t = t_0$ and **A**, **B**, **C**, and **D** are matrices of appropriate dimensions.

Similarity Transformations

It is readily seen that the definition of the state of a system is nonunique. Consider, for example, a linear transformation of $x(t)$ to $\overline{x}(t)$ defined as

$$\overline{x}(t) = \mathbf{T}^{-1}x(t) \qquad x(t) = \mathbf{T}\,\overline{x}(t)$$

where **T** is any nonsingular matrix, called a similarity transformation.

The following alternative state description is obtained

$$\dot{\overline{x}}(t) = \overline{\mathbf{A}}\,\overline{x}(t) + \overline{\mathbf{B}}\,u(t) \qquad \overline{x}(t_o) = \mathbf{T}^{-1}x_o$$
$$y(t) = \overline{\mathbf{C}}\,\overline{x}(t) + \overline{\mathbf{D}}\,u(t)$$

where

$$\overline{\mathbf{A}} \triangleq \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad \overline{\mathbf{B}} \triangleq \mathbf{T}^{-1}\mathbf{B} \quad \overline{\mathbf{C}} \triangleq \mathbf{C}\mathbf{T} \quad \overline{\mathbf{D}} \triangleq \mathbf{D}$$

The above model is an equally valid description of the system.

An illustration, say that the matrix **A** can be diagonalized by a similarity transformation **T**; then

$$\overline{\mathbf{A}} = \mathbf{\Lambda} \stackrel{\triangle}{=} \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

where if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of **A**, then

$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$$

Because Λ is diagonal, we have

$$\overline{x}_i(t) = e^{\lambda_i(t-t_o)}\overline{x}_o + \int_{t_o}^{t} e^{\lambda_i(t-\tau)}\overline{b}_i u(\tau)d\tau$$

where the subscript *i* denotes the *i*th component of the state vector.

Example

$$\mathbf{A} = \begin{bmatrix} -4 & -1 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix} \quad \mathbf{D} = 0$$

The matrix **T** can also be obtained by using the MATLAB command **eig**, which yields

$$\mathbf{T} = \begin{bmatrix} 0.8018 & 0.7071 & 0.0000 \\ 0.2673 & -0.7071 & 0.7071 \\ -0.5345 & -0.0000 & 0.7071 \end{bmatrix}$$

We obtain the similar state space description given by

$$\overline{\mathbf{A}} = \mathbf{\Lambda} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad \overline{\mathbf{B}} = \begin{bmatrix} 0.0 \\ -1.414 \\ 0.0 \end{bmatrix};$$

$$\overline{\mathbf{C}} = \begin{bmatrix} -0.5345 & -1.4142 & 0.7071 \end{bmatrix} \quad \overline{\mathbf{D}} = 0$$

Transfer Functions Revisited

The solution to the state equation model can be obtained via

$$\begin{aligned} Y(s) &= [\overline{\mathbf{C}}(s\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \overline{\mathbf{D}}]U(s) + \overline{\mathbf{C}}(s\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{x}(0) \\ &= [\mathbf{CT}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT})^{-1}\mathbf{T}^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{CT}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT})^{-1}\mathbf{T}^{-1}x(0) \\ &= [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \end{aligned}$$

We thus see that different choices of state variables lead to different internal descriptions of the model, but to the same input-output model, because the system transfer function can be expressed in either of the two equivalent fashions.

$$\overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\overline{\mathbf{D}}=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

for any nonsingular **T**.

From Transfer Function to State Space Representation

We have seen above how to go from a state space description to the corresponding transfer function. The converse operation leads to the following question:

> Given a transfer function $G(s)$, how can a state representation for this system be obtained?

Development

Consider a transfer function $G(s) = {}^{B(s)}/_{A(s)}$. We can then write

$$Y(s) = \sum_{i=1}^{n} b_{i-1} V_i(s) \qquad \text{where} \qquad V_i(s) = \frac{s^{i-1}}{A(s)} U(s)$$

We note from the above definitions that

$$v_i(t) = \mathcal{L}^{-1}\left[V(s)\right] = \frac{dv_{i-1}(t)}{dt} \qquad \text{for} \qquad i = 1, 2, \ldots, n$$

We can then choose, as state variables, $x_i(t) = v_i(t)$, which lead to the following state space model for the system.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} \qquad \mathbf{D} = 0$$

The above model has a special form. We will see later that any *completely controllable* system can be expressed in this way. Before we do this, we need to introduce the idea of controllability.

Controllability and Stabilizability

An important question that lies at the heart of control using state space models is whether we can steer the state via the control input to certain locations in the state space. Technically, this property is called controllability or reachability. A closely related issue is that of stabilizability. We will begin with controllability.

Controllability

The issue of controllability concerns whether a given initial state x_0 can be steered to the origin in finite time using the input $u(t)$.

Formally, we have the following:

Definition 17.1: A state x_0 is said to be controllable if there exists a finite interval $[0, T]$ and an input $\{u(t), t \in [0, T]\}$ such that $x(T) = 0$. If all states are controllable, then the system is said to be completely controllable.

Our next step will be to derive a simple algebraic test for controllability that can easily be applied to a given state space model. In deriving this result, we will use a result from linear algebra known as the Cayley-Hamilton Theorem.

Theorem 17.1: (*Cayley-Hamilton theorem*). Every matrix satisfies its own characteristic equation - i.e., if

$$\det(s\mathbf{I} - \mathbf{A}) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$$

then

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \ldots + a_0\mathbf{I} = 0$$

Proof: See the book.

Test for Controllability

Theorem 17.2: Consider the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

(*i*) The set of all controllable states is the range space of the controllability matrix Γ_c[**A**, **B**], where

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] \triangleq [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

(*ii*) The model is completely controllable if and only if where Γ_c[**A**, **B**] has full row rank.

Proof: Uses Cayley-Hamilton Theorem - see book.

Example

Consider the state space model

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by

$$\boldsymbol{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix}$$

Clearly, rank Γ_c[**A**, **B**] = 2; thus, the system is completely controllable.

Example

For

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The controllability matrix is given by:

$$\mathbf{\Gamma}_c[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Rank Γ_c[**A**, **B**] = 1 < 2; thus, the system is not completely controllable.

Although we have derived the above result by using the delta model, it holds equally for shift and/or continuous-time models.

We see that controllability is a black and white issue: a model either is completely controllable or it is not. Clearly, to know that something is uncontrollable is a valuable piece of information. However, to know that something is controllable really tells us nothing about the *degree* of controllability, i.e., about the difficulty that might be involved in achieving a certain objective. The latter issue lies at the heart of the fundamental design trade-offs in control that were the subject of Chapters 8 and 9.

Observability

Consider again the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

In general, the dimension of the observed output, *y*, can be less than the dimension of the state, *x*. However, one might conjecture that, if one observed the output over some nonvanishing time interval, then this might tell us something about the state. The associated properties are called observability (*or reconstructability*).

Observability

Observability is concerned with the issue of what can be said about the state when one is given measurements of the plant output.

A formal definition is as follows:

Definition : The state $x_0 \neq 0$ is said to be unobservable if, given $x(0) = x_0$, and $u[k] = 0$ for $k \geq 0$, then $y[k] = 0$ for $k \geq 0$. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

Test for Observability

A test for observability of a system is established in the following theorem.

Theorem : Consider the state model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

(*i*) The set of all unobservable states is equal to the null space of the observability matrix Γ_0[**A**, **C**], where

$$\mathbf{\Gamma}_o[\mathbf{A},\mathbf{C}] \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

(ii) The system is completely observable if and only if Γ_0[**A**, **C**], has full column rank *n*.

Summary

- State variables are system internal variables, upon which a full model for the system behavior can be built. The state variables can be ordered in a state vector.
- Given a linear system, the choice of state variables is not unique - however,
  - the minimal dimension of the state vector is a system invariant,
  - there exists a nonsingular matrix that defines a similarity transformation between any two state vectors, and
  - any designed system output can be expressed as a linear combination of the state variables and the inputs.

- For linear, time-invariant systems, the state space model is expressed in the following equations:

continuous-time systems

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

discrete-time systems, shift form

$$x[k+1] = \mathbf{A}_q x[k] + \mathbf{B}_q u[k]$$
$$y[k] = \mathbf{C}_q x[k] + \mathbf{D}_q u[k]$$

discrete-time systems, delta form

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

- Controllability tells us about the feasibility of attempting to control a plant.
- Observability tells us about whether it is possible to know what is happening inside a given system by observing its outputs.

- A transfer function can always be derived from a state space model.
- A state space model can be built from a transfer-function model. However, only the completely controllable and observable part of the system is described in that state space model. Thus *the transfer-function model might be only a partial description of the system*.

References

- Goodwin Graebe Salgado,Prentice Hall