Question Bank

Mathematics – III | **3rd Semester (Civil)** | **BTAM 301-18**

I. Find the gradient of the function f:

1. $f = (x + 1)(2y - 1)$
2. $f = 9x^2 + 4y^2$
3. $f = y/x$
4. $(y + 6)^2 + (x - 4)^2$
5. $f = x^4 + y^4$
6. $f = (x^2 - y^2)/(x^2 + y^2)$

II. Prove the following formulae:

$\nabla(f^n) = nf^{n-1}\nabla f$
$\nabla(fg) = f\nabla g + g\nabla f$
$\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$
$\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$

III. Find the gradient and its value at P:

$f = xy, \quad P: (-4, 5)$
$f = x/(x^2 + y^2), \quad P: (1, 1)$
$f = \ln(x^2 + y^2), \quad P: (8, 6)$
$f = (x^2 + y^2 + z^2)^{-1/2} \quad P: (12, 0, 16)$
$f = 4x^2 + 9y^2 + z^2, \quad P: (5, -1, -11)$

IV.

For what points $P: (x, y, z)$ does ∇f with $f = 25x^2 + 9y^2 + 16z^2$ have the direction from P to the origin

V.

1–6 CALCULATION OF THE DIVERGENCE

Find div **v** and its value at P.

1. $\mathbf{v} = x^2, 4y^2, 9z^2$, $P: (-1, 0, \frac{1}{2}$
2. $\mathbf{v} = 0, \cos xyz, \sin xyz$, $P: (2, \frac{1}{2}\pi, 0$
3. $\mathbf{v} = (x^2 + y^2)^{-1}\ x, y$
4. $\mathbf{v} = v_1(y, z), v_2(z, x), v_3(x, y)$, $P: (3, 1, -1)$
5. $\mathbf{v} = x^2y^2z^2\ x, y, z$, $P: (3, -1, 4)$
6. $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}\ x, y, z$
7. For what v_3 is $\mathbf{v} = e^x \cos y, e^x \sin y, v_3$ solenoidal
8. Let $\mathbf{v} = x, y, v_3$. Find a v_3 such that **(a)** div $\mathbf{v} > 0$ everywhere, **(b)** div $\mathbf{v} > 0$ if $|z| < 1$ and div $\mathbf{v} < 0$ if $|z| > 1$.

VI.

PROJECT. Useful Formulas for the Divergence. Prove

(a) $\text{div}\,(k\mathbf{v}) = k\,\text{div}\,\mathbf{v}$ (k constant)
(b) $\text{div}\,(f\mathbf{v}) = f\,\text{div}\,\mathbf{v} + \mathbf{v} \cdot \nabla f$
(c) $\text{div}\,(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$
(d) $\text{div}\,(f\nabla g) - \text{div}\,(g\nabla f) = f\nabla^2 g - g\nabla^2 f$

Verify (b) for $f = e^{xyz}$ and $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$. Obtain the answer to Prob. 6 from (b). Verify (c) for $f = x^2 - y^2$ and $g = e^{x+y}$. Give examples of your own for which (a)–(d) are advantageous.

VII.

15–20 LAPLACIAN

Calculate $\nabla^2 f$ by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

15. $f = \cos^2 x + \sin^2 y$

16. $f = e^{xyz}$

17. $f = \ln(x^2 + y^2)$

18. $f = z - \sqrt{x^2 + y^2}$

19. $f = 1/(x^2 + y^2 + z^2)$

20. $f = e^{2x} \cosh 2y$

VIII.

4–8 CALCULUTION OF CURL

Find curl $\mathbf{v}$ for $\mathbf{v}$ given with respect to right-handed Cartesian coordinates. Show the details of your work.

4. $\mathbf{v} = 2y^2, 5x, 0$

5. $\mathbf{v} = xyz \; x, y, z$

6. $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2} \; x, y, z$

7. $\mathbf{v} = 0, 0, \; e^{-x} \sin y$

8. $\mathbf{v} = \; e^{-z^2}, e^{-x^2}, e^{-y^2}$

IX.

PROJECT. Useful Formulas for the Curl. Assuming sufficient differentiability, show that

(a) curl $(\mathbf{u} + \mathbf{v})$ = curl $\mathbf{u}$ + curl $\mathbf{v}$

(b) div (curl $\mathbf{v}$) = 0

(c) curl $(f\mathbf{v})$ = (grad f) $\times \mathbf{v} + f$ curl $\mathbf{v}$

(d) curl (grad f) = $\mathbf{0}$

(e) div $(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot$ curl $\mathbf{u} - \mathbf{u} \cdot$ curl $\mathbf{v}$

X.

15–20 DIV AND CURL

With respect to right-handed coordinates, let $\mathbf{u} = \; y, z, x$, $\mathbf{v} = \; yz, zx, xy$, $f = xyz$, and $g = x + y + z$. Find the given expressions. Check your result by a formula in Proj. 14 if applicable.

15. curl $(\mathbf{u} + \mathbf{v})$, curl $\mathbf{v}$

16. curl $(g\mathbf{v})$

17. $\mathbf{v} \cdot$ curl $\mathbf{u}$, $\mathbf{u} \cdot$ curl $\mathbf{v}$, $\mathbf{u} \cdot$ curl $\mathbf{u}$

18. div $(\mathbf{u} \times \mathbf{v})$

19. curl $(g\mathbf{u} + \mathbf{v})$, curl $(g\mathbf{u})$

20. div (grad (fg))

XI.

2–11 LINE INTEGRAL. WORK

Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If **F** is a force, this gives the work done by the force in the displacement along C. Show the details.

2. $\mathbf{F} = \; y^2, -x^2 \;, \quad C\text{: } y = 4x^2$ from $(0, 0)$ to $(1, 4)$

3. **F** as in Prob. 2, C from $(0, 0)$ straight to $(1, 4)$. Compare.

4. $\mathbf{F} = \; xy, x^2y^2 \;, \quad C$ from $(2, 0)$ straight to $(0, 2)$

5. **F** as in Prob. 4, C the quarter-circle from $(2, 0)$ to $(0, 2)$ with center $(0, 0)$

6. $\mathbf{F} = \; x - y, y - z, z - x \;, \quad C\text{: } \mathbf{r} = \; 2\cos t, t, 2\sin t$ from $(2, 0, 0)$ to $(2, 2\pi, 0)$

7. $\mathbf{F} = \; x^2, y^2, z^2 \;, \quad C\text{: } \mathbf{r} = \; \cos t, \sin t, e^t$ from $(1, 0, 1)$ to $(1, 0, e^{2\pi})$. Sketch C.

8. $\mathbf{F} = \; e^x, \cosh y, \sinh z \;, \quad C\text{: } \mathbf{r} = \; t, t^2, t^3$ from $(0, 0, 0)$ to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$. Sketch C.

9. $\mathbf{F} = \; x + y, y + z, z + x \;, \quad C\text{: } \mathbf{r} = \; 2t, 5t, t$ from $t = 0$ to 1. Also from $t = -1$ to 1.

10. $\mathbf{F} = \; x, -z, 2y$ from $(0, 0, 0)$ straight to $(1, 1, 0)$, then to $(1, 1, 1)$, back to $(0, 0, 0)$

11. $\mathbf{F} = \; e^{-x}, e^{-y}, e^{-z} \;, \quad C\text{: } \mathbf{r} = \; t, t^2, t$ from $(0, 0, 0)$ to $(2, 4, 2)$. Sketch C.

XII.

15–20 INTEGRALS (8) AND (8*)

Evaluate them with **F** or f and C as follows.

15. $\mathbf{F} = \; y^2, z^2, x^2 \;, \quad C\text{: } \mathbf{r} = \; 3\cos t, 3\sin t, 2t \;, \; 0 \leqq t \leqq 4\pi$

16. $f = 3x + y + 5z, \quad C\text{: } \mathbf{r} = \; t, \cosh t, \sinh t \;, \; 0 \leqq t \leqq 1$. Sketch C.

17. $\mathbf{F} = \; x + y, y + z, z + x \;, \quad C\text{: } \mathbf{r} = \; 4\cos t, \sin t, 0 \;, \; 0 \leqq t \leqq \pi$

18. $\mathbf{F} = \; y^{1/3}, x^{1/3}, 0 \;, \quad C$ the hypocycloid $\mathbf{r} = \; \cos^3 t, \sin^3 t, 0 \;, \quad 0 \leqq t \leqq \pi/4$

19. $f = xyz, \quad C\text{: } \mathbf{r} = \; 4t, 3t^2, 12t \;, \quad -2 \leqq t \leqq 2$. Sketch C.

20. $\mathbf{F} = \; xz, yz, x^2y^2 \;, \quad C\text{: } \mathbf{r} = \; t, t, e^t \;, \quad 0 \leqq t \leqq 5$. Sketch C.

XIII.

3–9 PATH INDEPENDENT INTEGRALS

Show that the form under the integral sign is exact in the plane (Probs. 3–4) or in space (Probs. 5–9) and evaluate the integral. Show the details of your work.

3. $\displaystyle\int_{(\pi/2,\, \pi)}^{(\pi,\, 0)} (\tfrac{1}{2}\cos \tfrac{1}{2}x \cos 2y\, dx - 2\sin \tfrac{1}{2}x \sin 2y\, dy)$

4. $\displaystyle\int_{(4,\, 0)}^{(6,\, 1)} e^{4y}(2x\, dx + 4x^2\, dy)$

5. $\displaystyle\int_{(0,\, 0,\, \pi)}^{(2,\, 1/2,\, \pi/2)} e^{xy}(y\sin z\, dx + x\sin z\, dy + \cos z\, dz)$

6. $\displaystyle\int_{(0,\, 0,\, 0)}^{(1,\, 1,\, 0)} e^{x^2+y^2+z^2}(x\, dx + y\, dy + z\, dz)$

7. $\displaystyle\int_{(0,\, 2,\, 3)}^{(1,\, 1,\, 1)} (yz\sinh xz\, dx + \cosh xz\, dy + xy\sinh xz\, dz)$

XIV.

13–19 PATH INDEPENDENCE?

Check, and if independent, integrate from $(0, 0, 0)$ to (a, b, c).

13. $2e^{x^2}(x\cos 2y\, dx - \sin 2y\, dy)$

14. $(\sinh xy)\,(z\, dx - x\, dz)$

15. $x^2y\, dx - 4xy^2\, dy + 8z^2x\, dz$

16. $e^y\, dx + (xe^y - e^z)\, dy - ye^z\, dz$

17. $4y\, dx + z\, dy + (y - 2z)\, dz$

18. $(\cos xy)(yz\, dx + xz\, dy) - 2\sin xy\, dz$

19. $(\cos(x^2 + 2y^2 + z^2))\,(2x\, dx + 4y\, dy + 2z\, dz)$

XV.

1–10 LINE INTEGRALS: EVALUATION BY GREEN'S THEOREM

Evaluate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ counterclockwise around the boundary C of the region by Green's theorem, where

1. $\mathbf{F} = y, -x$, C the circle $x^2 + y^2 = 1/4$
2. $\mathbf{F} = 6y^2, 2x - 2y^4$, the square with vertices $\pm(2, 2), \pm(2, -2)$
3. $\mathbf{F} = x^2e^y, y^2e^x$, the rectangle with vertices (0, 0), (2, 0), (2, 3), (0, 3)
4. $\mathbf{F} = x \cosh 2y, 2x^2 \sinh 2y$, $: x^2 \leqq y \leqq x$
5. $\mathbf{F} = x^2 + y^2, x^2 - y^2$, $: 1 \leqq y \leqq 2 - x^2$
6. $\mathbf{F} = \cosh y, -\sinh x$, $: 1 \leqq x \leqq 3, x \leqq y \leqq 3x$
7. $\mathbf{F} = \text{grad}\,(x^3 \cos^2 (xy))$, as in Prob. 5
8. $\mathbf{F} = -e^{-x} \cos y, -e^{-x} \sin y$, the semidisk $x^2 + y^2 \leqq 16, x \geqq 0$
9. $\mathbf{F} = e^{y/x}, e^y \ln x + 2x$, $: 1 + x^4 \leqq y \leqq 2$
10. $\mathbf{F} = x^2y^2, -x/y^2$, $: 1 \leqq x^2 + y^2 \leqq 4, x \geqq 0, y \geqq x$. Sketch .

XVI.

1–10 FLUX INTEGRALS (3) $\int_S \mathbf{F} \cdot \mathbf{n}\, dA$

Evaluate the integral for the given data. Describe the kind of surface. Show the details of your work.

1. $\mathbf{F} = -x^2, y^2, 0$, S: $\mathbf{r} = u, v, 3u - 2v$, $0 \leqq u \leqq 1.5, \quad -2 \leqq v \leqq 2$
2. $\mathbf{F} = e^y, e^x, 1$, S: $x + y + z = 1, \quad x \geqq 0, \quad y \geqq 0, \quad z \geqq 0$
3. $\mathbf{F} = 0, x, 0$, S: $x^2 + y^2 + z^2 = 1, \quad x \geqq 0, \quad y \geqq 0, \quad z \geqq 0$
4. $\mathbf{F} = e^y, -e^z, e^x$, S: $x^2 + y^2 = 25, \quad x \geqq 0, \quad y \geqq 0, \quad 0 \leqq z \leqq 2$
5. $\mathbf{F} = x, y, z$, S: $\mathbf{r} = u \cos v, u \sin v, u^2$, $0 \leqq u \leqq 4, \quad -\pi \leqq v \leqq \pi$
6. $\mathbf{F} = \cosh y, 0, \sinh x$, S: $z = x + y^2, \quad 0 \leqq y \leqq x, \quad 0 \leqq x \leqq 1$
7. $\mathbf{F} = 0, \sin y, \cos z$, S the cylinder $x = y^2$, where $0 \leqq y \leqq \pi/4$ and $0 \leqq z \leqq y$
8. $\mathbf{F} = \tan xy, x, y$, S: $y^2 + z^2 = 1, \quad 2 \leqq x \leqq 5, \quad y \geqq 0, \quad z \geqq 0$
9. $\mathbf{F} = 0, \text{s inh}\, z, \cosh x$, S: $x^2 + z^2 = 4$, $0 \leqq x \leqq 1/\sqrt{2}, \quad 0 \leqq y \leqq 5, \quad z \geqq 0$
10. $\mathbf{F} = y^2, x^2, z^4$, S: $z = 4\sqrt{x^2 + y^2}, \quad 0 \leqq z \leqq 8, \quad y \geqq 0$

XVII.

9–18 APPLICATION OF THE DIVERGENCE THEOREM

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$ by the divergence theorem. Show the details.

9. $\mathbf{F} = x^2, 0, z^2$, S the surface of the box $|x| \leqq 1$, $|y| \leqq 3$, $0 \leqq z \leqq 2$

10. Solve Prob. 9 by direct integration.

11. $\mathbf{F} = e^x, e^y, e^z$, S the surface of the cube $|x| \leqq 1$, $|y| \leqq 1$, $|z| \leqq 1$

12. $\mathbf{F} = x^3 - y^3, y^3 - z^3, z^3 - x^3$, S the surface of $x^2 + y^2 + z^2 \leqq 25$, $z \geqq 0$

13. $\mathbf{F} = \sin y, \cos x, \cos z$, S, the surface of $x^2 + y^2 \leqq 4$, $|z| \leqq 2$ (a cylinder and two disks)

14. $\mathbf{F}$ as in Prob. 13, S the surface of $x^2 + y^2 \leqq 9$, $0 \leqq z \leqq 2$

15. $\mathbf{F} = 2x^2, \frac{1}{2}y^2, \sin \pi z$, S the surface of the tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)

16. $\mathbf{F} = \cosh x, z, y$, S as in Prob. 15

17. $\mathbf{F} = x^2, y^2, z^2$, S the surface of the cone $x^2 + y^2 \leqq z^2$, $0 \leqq z \leqq h$

18. $\mathbf{F} = [xy, yz, zx$, S the surface of the cone $x^2 + y^2 \leqq 4z^2$, $0 \leqq z \leqq 2$

XVIII.

1–10 DIRECT INTEGRATION OF SURFACE INTEGRALS

Evaluate the surface integral $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dA$ directly for the given $\mathbf{F}$ and S.

1. $\mathbf{F} = z^2, -x^2, 0$, S the rectangle with vertices (0, 0, 0), (1, 0, 0), (0, 4, 4), (1, 4, 4)

2. $\mathbf{F} = -13 \sin y, 3 \sinh z, x$, S the rectangle with vertices (0, 0, 2), (4, 0, 2), $(4, \pi/2, 2)$, $(0, \pi/2, 2)$

3. $\mathbf{F} = e^{-z}, e^{-z} \cos y, e^{-z} \sin y$, S: $z = y^2/2$, $-1 \leqq x \leqq 1$, $0 \leqq y \leqq 1$

4. $\mathbf{F}$ as in Prob. 1, $z = xy$ $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 4)$. Compare with Prob. 1.

5. $\mathbf{F} = z^2, \frac{3}{2}x, 0$, S: $0 \leqq x \leqq a$, $0 \leqq y \leqq a$, $z = 1$

6. $\mathbf{F} = y^3, -x^3, 0$, S: $x^2 + y^2 \leqq 1$, $z = 0$

7. $\mathbf{F} = e^y, e^z, e^x$, S: $z = x^2$ $(0 \leqq x \leqq 2,$ $0 \leqq y \leqq 1)$

8. $\mathbf{F} = z^2, x^2, y^2$, S: $z = \sqrt{x^2 + y^2}$, $y \geqq 0$, $0 \leqq z \leqq h$

9. Verify Stokes's theorem for $\mathbf{F}$ and S in Prob. 5.

10. Verify Stokes's theorem for $\mathbf{F}$ and S in Prob. 6.

XIX.

In each of Exercises 1–8, use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ for the given $f(t)$.

1. $f(t) = t^2$.

2. $f(t) = \sinh t$.

3. $f(t) = \begin{cases} 5, & 0 < t < 2, \\ 0, & t > 2. \end{cases}$

4. $f(t) = \begin{cases} 4, & 0 < t < 3, \\ 2, & t > 3. \end{cases}$

5. $f(t) = \begin{cases} t, & 0 < t < 2, \\ 3, & t > 2. \end{cases}$

6. $f(t) = \begin{cases} 0, & 0 < t < 1, \\ t, & 1 < t < 2, \\ 1, & t > 2. \end{cases}$

7. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$

8. $f(t) = \begin{cases} 2t, & 0 \le t < 1, \\ 2, & 1 \le t < 3, \\ 8 - 2t, & t \ge 3. \end{cases}$

XX.

1. Use Theorem 9.2 to find $\mathcal{L}\{\cos^2 at\}$.

2. Use Theorem 9.2 to find $\mathcal{L}\{\sin at \sin bt\}$.

3. Use Theorem 9.2 to find $\mathcal{L}\{\sin^3 at\}$ and then employ Theorem 9.3 to obtain $\mathcal{L}\{\sin^2 at \cos at\}$.

4. Use Theorem 9.2 to find $\mathcal{L}\{\cos^3 at\}$ and then employ Theorem 9.3 to obtain $\mathcal{L}\{\cos^2 at \sin at\}$.

5. If $\mathcal{L}\{t^2\} = 2/s^3$, use Theorem 9.3 to find $\mathcal{L}\{t^3\}$.

6. If $\mathcal{L}\{t^2\} = 2/s^3$, use Theorem 9.4 to find $\mathcal{L}\{t^4\}$.

7. Use (9.11) and (9.13) to find $\mathcal{L}\{f(t)\}$ if

$$f''(t) + 3f'(t) + 2f(t) = 0, \quad f(0) = 1, \quad \text{and} \quad f'(0) = 2.$$

8. Use (9.11) and (9.13) to find $\mathcal{L}\{f(t)\}$ if

$$f''(t) + 4f'(t) - 8f(t) = 0, \quad f(0) = 3, \quad f'(0) = -1.$$

9. Use formulas (9.17) and (9.11) to find $\mathcal{L}\{f(t)\}$ if

$$f'''(t) = f'(t),$$
$$f''(0) = 2, \quad f'(0) = 1, \quad \text{and} \quad f(0) = 0.$$

10. Use formulas (9.17) and (9.18) to find $\mathcal{L}\{f(t)\}$ if

$$f^{iv}(t) = f''(t),$$
$$f'''(0) = 1, \quad f''(0) = 0, \quad f'(0) = 0, \quad \text{and} \quad f(0) = -1.$$

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XXI.

11. Use formulas (9.11) and (9.18) and Example 9.3 to find $\mathcal{L}\{f(t)\}$ if

$$2f''(t) + 3f'(t) + 4f(t) = e^{5t},$$
$$f(0) = -3, \quad \text{and} \quad f'(0) = 2.$$

12. Use formulas (9.11) and (9.18) and Example 9.4 to find $\mathcal{L}\{f(t)\}$ if

$$3f''(t) - 5f'(t) + 7f(t) = \sin 2t,$$
$$f(0) = 4, \quad \text{and} \quad f'(0) = 6.$$

13. Use Theorem 9.5 to find $\mathcal{L}\{e^{at}t^2\}$.

14. Use Theorem 9.5 to find $\mathcal{L}\{e^{at} \sin^2 bt\}$.

15. Use Theorem 9.6 to find $\mathcal{L}\{t^2 \cos bt\}$.

16. Use Theorem 9.6 to find $\mathcal{L}\{t^3 \sin bt\}$.

17. Use Theorem 9.6 to find $\mathcal{L}\{t^3e^{at}\}$.

18. Use Theorem 9.6 to find $\mathcal{L}\{t^4e^{at}\}$.

XXII.

Use Table 9.1 to find $\mathcal{L}^{-1}\{F(s)\}$ for each of the functions F defined in Exercises 1–30.

1. $F(s) = \frac{2}{s} + \frac{3}{s-5}$.

2. $F(s) = \frac{4}{s+2} + \frac{7}{s}$.

3. $F(s) = \frac{2}{s^2+9}$.

4. $F(s) = \frac{2s}{s^2+9}$.

5. $F(s) = \frac{5}{(s-2)^4}$.

6. $F(s) = \frac{5s+6}{s^3}$.

7. $F(s) = \frac{s+2}{s^2+4s+7}$.

8. $F(s) = \frac{s+10}{s^2+8s+20}$.

9. $F(s) = \frac{3s}{s^2-4}$.

10. $F(s) = \frac{2s+3}{s^2-4}$.

11. $F(s) = \frac{s-2}{s^2+5s+6}$.

12. $F(s) = \frac{2s+6}{8s^2-2s-3}$.

13. $F(s) = \frac{5s}{s^2+4s+4}$.

14. $F(s) = \frac{s+1}{s^3+2s}$.

15. $F(s) = \frac{5}{(s+2)^5}$.

16. $F(s) = \frac{2s+7}{(s+3)^4}$.

17. $F(s) = \frac{7}{(2s+1)^3}$.

18. $F(s) = \frac{8(s+1)}{(2s+1)^3}$.

19. $F(s) = \frac{s+3}{(s^2+4)^2}$.

20. $F(s) = \frac{s^2-4s-4}{(s^2+4)^2}$.

21. $F(s) = \frac{2s+12}{s^2+6s+13}$.

22. $F(s) = \frac{5s+17}{s^2+4s+13}$.

23. $F(s) = \frac{10s+23}{s^2+7s+12}$.

24. $F(s) = \frac{s+7}{2s^2+s-1}$.

25. $F(s) = \frac{1}{s^3+4s^2+3s}$.

26. $F(s) = \frac{s+5}{s^4+3s^3+2s^2}$.

27. $F(s) = \frac{7s^2+8s+8}{s^3+4s}$.

28. $F(s) = \frac{3s^3+4s^2-16s+16}{s^3(s-2)^2}$.

29. $F(s) = \frac{s^3+16s}{(s^2+4)^2}$.

30. $F(s) = \frac{5s^2-18s+9}{s^4+18s^2+81}$.

XXIII.

In each of Exercises 1–6 find $\mathcal{L}^{-1}\{H(s)\}$ using the convolution and Table 9.1.

1. $H(s) = \frac{1}{s^2+5s+6}$.

2. $H(s) = \frac{1}{s^2+3s-4}$.

3. $H(s) = \frac{1}{s(s^2+9)}$.

4. $H(s) = \frac{1}{s(s^2+4s+13)}$.

5. $H(s) = \frac{1}{s^2(s+3)}$.

6. $H(s) = \frac{1}{(s+2)(s^2+1)}$.

XXIV.

Use the Laplace transforms to solve each of the initial-value problems in Exercises 1–22.

1. $y' - y = e^{3t}$,
$y(0) = 2$,

2. $y' + y = 2\sin t$,
$y(0) = -1$.

3. $y' + 4y = 6e^{-t}$,
$y(0) = 5$.

4. $y' + 2y = 16t^2$,
$y(0) = 7$.

5. $y'' - 5y' + 6y = 0$,
$y(0) = 1, \quad y'(0) = 2$.

6. $y'' + y' - 12y = 0$,
$y(0) = 4, \quad y'(0) = -1$.

7. $y'' - 6y' + 9y = 0$,
$y(0) = 2, \quad y'(0) = 9$.

8. $y'' + 2y' + 5y = 0$,
$y(0) = 2, \quad y'(0) = 4$.

9. $y'' + 4y = 8$,
$y(0) = 0, \quad y'(0) = 6$.

10. $y'' + 9y = 36e^{-3t}$,
$y(0) = 2, \quad y'(0) = 3$.

11. $y'' + 6y' + 8y = 16$,
$y(0) = 0, \quad y'(0) = 10$.

12. $2y'' + y' = 5e^{2t}$,
$y(0) = 2, \quad y'(0) = 0$.

13. $y^{iv} - y = 0$,
$y(0) = 0, \quad y'(0) = 1$,
$y''(0) = 1, \quad y'''(0) = 0$.

14. $y^{iv} - 2y'' + y = 0$,
$y(0) = 0, \quad y'(0) = 4$,
$y''(0) = 0, \quad y'''(0) = 8$.

15. $y'' - y' - 2y = 18e^{-t}\sin 3t$,
$y(0) = 0, \quad y'(0) = 3$.

16. $y'' + 2y' + y = te^{-2t}$,
$y(0) = 1, \quad y'(0) = 0$.

17. $y'' + 7y' + 10y = 4te^{-3t}$,
$y(0) = 0, \quad y'(0) = -1$.

18. $y'' - 8y' + 15y = 9te^{2t}$,
$y(0) = 5, \quad y'(0) = 10$.

19. $y'' + 3y' + 2y = 10\cos t$,
$y(0) = 0, \quad y'(0) = 7$.

20. $y'' + 5y' + 4y = (6t + 8)e^{-t}$,
$y(0) = 1, \quad y'(0) = 1$.

21. $y''' - 5y'' + 7y' - 3y = 20\sin t$,
$y(0) = 0, \quad y'(0) = 0$,
$y''(0) = -2$.

22. $y''' - 6y'' + 11y' - 6y = 36te^{4t}$,
$y(0) = -1, \quad y'(0) = 0$,
$y''(0) = -6$.

XXV.

Use Table 9.1 to find $\mathcal{L}^{-1}\{F(s)\}$ for each of the functions F defined in Exercises 1–14.

1. $F(s) = \dfrac{4s^2 + 6}{s^3} e^{-3s}$

2. $F(s) = \dfrac{3s + 1}{(s - 2)^2} e^{-5s}$

3. $F(s) = \dfrac{s}{s^2 - 5s + 6} e^{-2s}$

4. $F(s) = \dfrac{12}{s^2 + s - 2} e^{-4s}$

5. $F(s) = \dfrac{5s + 6}{s^2 + 9} e^{-\pi s}$.

6. $F(s) = \dfrac{s + 10}{s^2 + 2s - 8} e^{-2s}$.

7. $F(s) = \dfrac{s + 8}{s^2 + 4s + 13} e^{-(\pi s)/2}$.

8. $F(s) = \dfrac{2s + 9}{s^2 + 4s + 13} e^{-3s}$.

9. $F(s) = \dfrac{e^{-4s} - e^{-7s}}{s^2}$.

10. $F(s) = \dfrac{e^{-3s} - e^{-8s}}{s^3}$.

11. $F(s) = \dfrac{1 + e^{-\pi s}}{s^2 + 4}$.

12. $F(s) = \dfrac{2 - e^{-3s}}{s^2 + 9}$.

13. $F(s) = \dfrac{2[1 + e^{-(\pi s)/2}]}{s^2 - 2s + 5}$.

14. $F(s) = \dfrac{4(e^{-2s} - 1)}{s(s^2 + 4)}$.

XXVI.

Use Laplace transforms to solve each of the initial-value problems in Exercises 1–12.

1. $y' + 2y = h(t)$, where $h(t) = \begin{cases} 4, & 0 < t < 6, \\ 0, & t > 6, \end{cases}$

$y(0) = 5$.

2. $3y' - 5y = h(t)$, where $h(t) = \begin{cases} 0, & 0 < t < 6, \\ 10, & t > 6, \end{cases}$

$y(0) = 4$.

3. $y'' - 3y' + 2y = h(t)$, where $h(t) = \begin{cases} 2, & 0 < t < 4, \\ 0, & t > 4, \end{cases}$

$y(0) = 0, \quad y'(0) = 0$.

4. $y'' + 5y' + 6y = h(t)$, where $h(t) = \begin{cases} 6, & 0 < t < 2, \\ 0, & t > 2, \end{cases}$

$y(0) = 0, \quad y'(0) = 0$.

XXVII.

Use Laplace transforms to solve each of the initial-value problems in Exercises 1–6.

1. $y' - 4y = \delta(t - 2), \quad y(0) = 3$.

2. $y'' + 4y' + 5y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0$.

3. $y'' + y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1$.

4. $y'' + 3y' + 2y = \delta(t - 4), \quad y(0) = 2, \quad y'(0) = -6$.

5. $y'' + 4y' + 3y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = -3$.

6. $y'' + 4y' + 5y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = -2$.

XXVIII.

6–10 GRAPHS OF 2π–PERIODIC FUNCTIONS

Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$

7. $f(x) = |\sin x|, \quad f(x) = \sin |x|$

8. $f(x) = e^{-|x|}, \quad f(x) = |e^{-x}|$

9. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$

10. $f(x) = \begin{cases} -\cos^2 x & \text{if } -\pi < x < 0 \\ \cos^2 x & \text{if } 0 < x < \pi \end{cases}$

XXIX.

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

12. $f(x)$ in Prob. 6

13. $f(x)$ in Prob. 9

14. $f(x) = x^2 \quad (-\pi < x < \pi)$

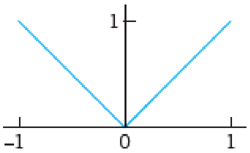
15. $f(x) = x^2 \quad (0 < x < 2\pi)$

XXX.

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd Find its Fourier series. Show details of your work.

8.



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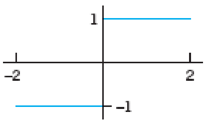

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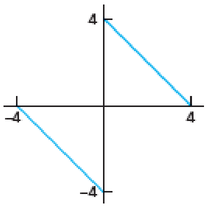
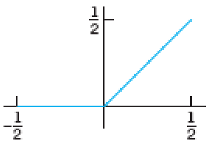


11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

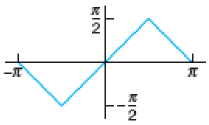
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

13.



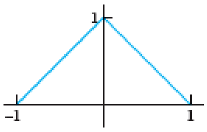
14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17.



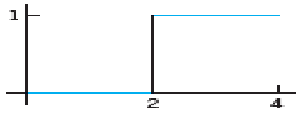
XXXI.

23–29 HALF-RANGE EXPANSIONS

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

23.

24.



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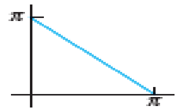

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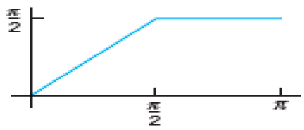

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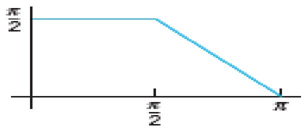

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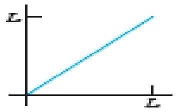

29. $f(x) = \sin x \ (0 < x < \pi)$

XXXII.

1–6 EVALUATION OF INTEGRALS

Show that the integral represents the indicated function. *Hint.* Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

1. $$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dx = \begin{cases} 0 & \text{if } x < 0 \\ \pi\ 2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\, dw = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi \end{cases}$$

3. $$\int_0^\infty \frac{1 - \cos \pi w}{w} \sin xw\, dw = \begin{cases} \frac{1}{2}\pi & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1 - w^2} \cos xw\, dw = \begin{cases} \frac{1}{2}\pi \cos x & \text{if } 0 < |x| < \frac{1}{2}\pi \\ 0 & \text{if } |x| \ge \frac{1}{2}\pi \end{cases}$$

5. $$\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin xw\, dw = \begin{cases} \frac{1}{2}\pi x & \text{if } 0 < x < 1 \\ \frac{1}{4}\pi & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

6. $$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}\, dw = \frac{1}{2}\pi e^{-x} \cos x \quad \text{if } x > 0$$

XXXIII.

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (10).

7. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

9. $f(x) = 1/(1 + x^2) \quad x > 0.$ *Int.* See (13).

10. $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

XXXIV.

16–20 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

16. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

17. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

19. $f(x) = \begin{cases} e^{x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

20. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

XXXV.

1–8 FOURIER COSINE TRANSFORM

1. Find the cosine transform $\hat{f}_c(w)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.

2. Find f in Prob. 1 from the answer $\hat{f}_c$.

3. Find $\hat{f}_c(w)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.

4. Derive formula 3 in Table I of Sec. 11.10 by integration.

5. Find $\hat{f}_c(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

6. Continuity assumptions. Find $\hat{g}_c(w)$ for $g(x) = 2$ if $0 < x < 1$, $g(x) = 0$ if $x > 1$. Try to obtain from it $\hat{f}_c(w)$ for $f(x)$ in Prob. 5 by using (5a).

7. Existence? Does the Fourier cosine transform of $x^{-1} \sin x$ $(0 < x < \infty)$ exist Of $x^{-1} \cos x$ Give reasons.

8. Existence? Does the Fourier cosine transform of $f(x) = k = \text{const}$ $(0 < x < \infty)$ exist The Fourier sine transform

9–15 FOURIER SINE TRANSFORM

9. Find $\mathcal{F}_s(e^{-ax})$, $a > 0$, by integration.

10. Obtain the answer to Prob. 9 from (5b).

11. Find $\hat{f}_s(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

12. Find $\mathcal{F}_s(xe^{-x^2/2})$ from (4b) and a suitable formula in Table I of Sec. 11.10.

13. Find $\mathcal{F}_s(e^{-x})$ from (4a) and formula 3 of Table I in Sec. 11.10.

14. Gamma function. Using formulas 2 and 4 in Table II of Sec. 11.10, prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (30) in App. A3.1, a value needed for Bessel functions and other applications.

15. WRITING PROJECT. Finding Fourier Cosine and Sine Transforms. Write a short report on ways of obtaining these transforms, with illustrations by examples of your own.

XXXVI.

2–11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

References:

1. Advanced Engineering Mathematics by Erwin Ereyszig (Tenth Edition)

2. Introduction to Ordinary Differential Equations by Shepley L. Ross (4^{th} Edition)