# *Ordinary Differential Equations*

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# Syllabus

The syllabus contains the following articles:

- First Order Differential Equation
  - Leibnitz linear equation
  - Bernoulli's equation
  - Exact differential equation
  - Equations not of first degree
    - Equation solvable for $p$
    - Equation solvable for $x$
    - Equation solvable for $y$
  - Clairaut's equation
- Higher Order Differential Equation
  - Second order linear differential equations with variable coefficients
  - Method of variation of parameters
  -

# Leibnitz linear equation

## Definition

An equation of the form $\frac{dy}{dx} + Py = Q$, where $P$ and $Q$ are either constants or functions of $x$ only is called Leibnitz linear equation.
Alternately, the equation may be of the form $\frac{dx}{dy} + Px = Q$, where $P$ and $Q$ are either constants or functions of $y$ only.

## Solution

This equation is solved by evaluating the Integration Factor that is given by $IF = e^{\int P dx}$ and the solution is obtained by $y(IF) = \int Q(IF)dx + c$ for the former case and for the latter $x$ is replaced by $y$ in the IF and the solution.

# Questions

- $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$
- $x \log x \frac{dy}{dx} + y = 2 \log x$
- $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$
- $\frac{dy}{dx} = \frac{y}{2y \log y + y - x}$
- $\sqrt{1 - y^2}dx = (\sin^{-1} y - x)dy$

# Bernoulli's Equation

### Definition

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where $P$ and $Q$ are either constants or functions of $x$ only is called Bernoulli's equation.
Alternately, the equation may also be written as $\frac{dx}{dy} + Px = Qx^n$, where $P$ and $Q$ are either constants or functions of $y$ only.

### Solution

This equation is reduced to Leibnitz linear equation by substituting $y^{1-n} = z$ and differentiating. This generates the Leibnitz equation in $z$ and $x$ that is solved as explained earlier and then $z$ is resubstituted in terms of $y$. The corresponding changes are made in the latter case of definition.

# QUESTIONS

- $x\dfrac{dy}{dx} + y = x^3y^6$
- $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$
- $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$
- $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$
- $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

# Exact Differential Equation

## Definition

An equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be an Exact differntial equation if it can be obtained directly by differentiating the equation $u(x, y) = c$, which is its primitive.
i.e. if

$$du = Mdx + Ndy$$

## Necessary and Sufficient Condition

The necessary and sufficient condition for the equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

## Solution

The solution of $Mdx + Ndy = 0$ is given by

$$\int_{y \text{ constant}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$$

# Questions

- $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$
- $(1 + e^{x/y})dx + \left(1 - \frac{x}{y}\right) e^{x/y} dy = 0$
- $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$
- $xdy + ydx + \frac{xdy - ydx}{x^2 + y^2} = 0$
- $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

# Equations reducible to exact equations

## Reducible to exact equations

Equations which are not exact can sometimes be made exact after multiplying by a suitable factor (function of $x$ and/or $y$) called the Integration Factor (IF).

## IF by Inspection

- $ydx + xdy = d(xy)$
- $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$
- $\dfrac{xdy - ydx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
- $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right]$
- $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$
- $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$
- $\dfrac{ydx + xdy}{xy} = d[\log(xy)]$
- $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

# Equations reducible to exact equations

### IF for Homoegeneous Equation

If $Mdx + Ndy = 0$ is a Homoegeneous equation in $x$ and $y$, then $\dfrac{1}{Mx + Ny}$ is an IF provided $Mx + Ny \neq 0$.

### IF for $f_1(xy)ydx + f_2(xy)xdy = 0$

For equation of this type, IF is given by $\dfrac{1}{Mx - Ny}$.

# Equations reducible to exact equations

## IF for $Mdx + Ndy = 0$

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of $x$ only, say $f(x)$, then $IF = e^{\int f(x)dx}$.
- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of $y$ only, say $g(y)$, then $IF = e^{\int g(y)dy}$.

## IF for $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$

In this equation, $a, b, c, d, m, n, p, q$ are all constants and IF is given by $x^h y^k$, where $h$ and $k$ are so chosen that the equation becomes exact after multiplication with IF.

# QUESTIONS

- $(1+xy)ydx+(1-xy)xdy=0$
- $xdy-ydx=xy^2dx$
- $(xye^{x/y}+y^2)dx-x^2e^{x/y}dy=0$
- $(x^2y^2+xy+1)ydx+(x^2y^2-xy+1)xdy=0$
- $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)dy=0$
- $(2x^2y-3y^4)dx+(3x^3+2xy^3)dy=0$
- $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$

# Equations of first order and higher degree

### Definition

A differential equation of the first order and $n^{th}$ degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0, \text{ where } p = \frac{dy}{dx} \quad (1)$$

# Equations solvable for $p$

Resolve equation (1) into $n$ linear factors and solve each of the factors to obtain solution of the given equation.

## Questions

- $p^2 - 7p + 12 = 0$
- $xyp^2 - (x^2 + y^2)p + xy = 0$
- $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$
- $p^2 - 2p\sinh x - 1 = 0$
- $4y^2p^2 + 2pxy(3x + 1)3x^3 = 0$

# Equations solvable for $y$

Differentiate equation (1), wrt $x$, to obtain a differential equation of first order in $p$ and $x$ that has solution of the form $\phi(x,p,c)=0$. The elimination $p$ from this solution and equation (1) gives the desired solution.

## Questions

- $xp^2 - 2yp + ax = 0$
- $y - 2px = \tan^{-1}(xp^2)$
- $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$
- $x - yp = ap^2$

# Equations solvable for $x$

Differentiate equation (1), wrt $y$, to obtain a differential equation of first order in $p$ and $y$ that has solution of the form $\phi(y,p,c)=0$. The elimination $p$ from this solution and equation (1) gives the desired solution.

## Questions

- $y = 3px + 6p^2y^2$
- $p^3 - 4xyp + 8y^2 = 0$
- $y = 2px + p^2y$
- $y^2 \log y = xyp + p^2$

# Clairaut's equation

## Definition

An equation of the form $y = px + f(p)$ is called Clairaut's equation.

## Solution

Differente the equation wrt $x$, and obtain the solution by putting $p = c$ in the given equation.

## Questions

- $y = xp + \dfrac{a}{p}$
- $y = px + \sqrt{a^2p^2 + b^2}$
- $p = \sin(y - px)$
- $p = \log(px - y)$

# Linear Differential Equations

### Definition

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the $n^{th}$ order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X \tag{2}$$

# Linear Differential Equations

## Complementary Function (CF)

- If all the roots of equation (2) are real and distint, CF is given by $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
- If two roots are equal, say $m_1 = m_2$, then CF is given by $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
- If two roots are imaginary, say $m_1 = \alpha + \iota\beta,\ m_2 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
- It two pairs of imaginary roots are equal, say $m_1 = m_2 = \alpha + \iota\beta,\ m_3 = m_4 = \alpha - \iota\beta$, then CF is given by $y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

# Linear Differential Equations

## Particular Integral (PI)

- If $X = e^{ax}$, then PI is given by $y = \dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$, provided $f(a) \neq 0$.
- If $X = \sin(ax+b)$ or $\cos(ax+b)$, then PI is given by $y = \dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)$. Likewise for $\cos(ax+b)$.
- If $X = x^m$, where $m$ is a positive integer, then PI is given by $y = \dfrac{1}{(D)}x^m$.

  Take out the lowest degree term from $f(D)$ to make the first term unity and then shift the remaining term to numerator and apply Binomial expansion upto $D^m$. Operate term by term on $x^m$.
- If $X = e^{ax}V$, where $V$ is a function of $x$, then PI is given by $y = \dfrac{1}{f(D)}e^{ax}V = e^{ax}\dfrac{1}{f(D+a)}V$.
- If $X$ is any other function of $x$, then PI is obtained by resolving the $f(D)$ into linear factors and applying $\dfrac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx$

# QUESTIONS

- $(D^2 + 4D + 5)y = -2\cosh x$
- $(D2 - 4D + 3)y = \sin 3x \cos 2x$
- $(D^2 + 4)y = e^x + \sin 2x$
- $(D^2 + D)y = x^2 + 2x + 4$
- $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$
- $(D^2 - 4D + 4)y = 8x^2e^{2x} \sin 2x$
- $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$
- $(D - 1)^2(D + 1)^2y = \sin^2 \frac{x}{2} + e^x + x$

# Cauchy's Homogeneous Equation

## Definition

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \qquad (3)$$

where $a_i$s are constants and $X$ is a function of $x$ is called Cauchy's Homegeneous Linear Equation.

## Solution

The equation is reduced to an LDE with constant coefficients by putting $z = e^x$ thereby generating an LDE in $x$ and $z$ that can be solved as explained earlier and finally the solution of equation (3) is obtained by putting $z = \log x$.

# QUESTIONS

- $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} - 25y = 50$
- $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$
- $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$
- $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x\dfrac{\sin(\log x) + 1}{x}$

# Legendre's Linear Equation

## Definition

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X \quad (4)$$

where $a_i$s, $a$ and $b$ are constants and $X$ is a function of $x$ is called Legendre's Linear Equation.

## Solution

The equation is reduced to an LDE with constant coefficients by putting $a+bx=e^z$ thereby generating an LDE in $x$ and $z$ that can be solved as explained earlier and finally the solution of equation (4) is obtained by putting $z=\log(a+bx)$.

# QUESTIONS

- $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$
- $(1+2x)^2\dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$
- $(3+2x)^2\dfrac{d^2y}{dx^2} - 2(3+2x)\dfrac{dy}{dx} - 12y = 6x$

# Variation of Parameters

This method is applicable for the second order differential equation of the form $\dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_2y = X$
Let the CF of this equation be

$$y = c_1y_1 + c_2y_2$$

. Then the PI of this equation is given by

$$y = uy_1 + vy_2$$

where

$$u = -\int \frac{y_2X}{W}dx$$

and

$$v = \int \frac{y_1X}{W}dx$$

where $W$ is the Wronskian of $y_1,\ y_2$.

# QUESTIONS

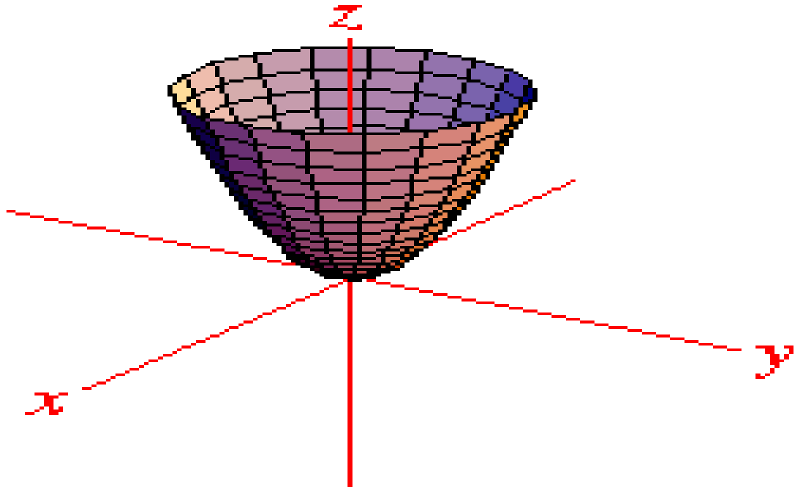
- $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$
- $\dfrac{d^2y}{dx^2} + y = cosec\ x$
- $\dfrac{d^2y}{dx^2} + y = x\sin x$
- $y'' - 2y' + 2y = e^x \tan x$

# PARTIAL DIFFERENTIATION

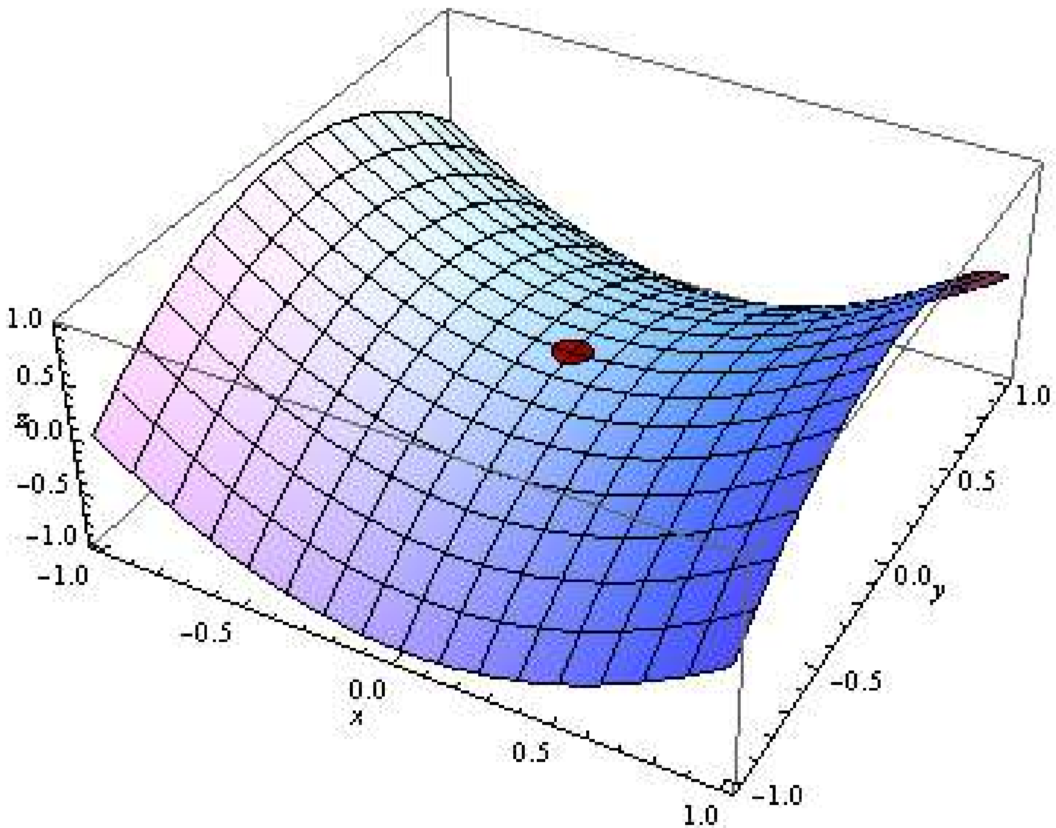
IMPORTANT TERMS, DEFINITIONS & RESULTS

## *01. Function of Two Variables*:

Let us consider a set of points D in the plane. A rule which assigns a unique real value $z$ to each point in D is called a real valued function defined on D. Using rectangular coordinates, we can identify every point on the plane by an ordered pair $(x, y)$ of real numbers. Thus the real valued function may be represented as $z = f(x, y)$, where $x$ and $y$ are called independent variables and $z$ is the dependent variable. D is known as the domain of the function $f$.




$z = x^2 + y^2$


1.0
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0.0 y
0.5
1.0

Similarly, when we hold $x$ equal to a constant $x_0$, $z=f(x,y)$ becomes the function $z=f(x_0,y)$ of $y$, whose graph is the intersection of the surface with the plane $x=x_0$ (Figure 2), and the $y-$derivative $f_y(x_0,y_0)$ is the slope in the positive $y-$direction of the tangent line to this curve at $y=y_0$.

*06. Higher Order Partial Derivatives:*

If the partial derivative $f_x$ exists at every point of a region D, then $f_x$ itself is a well defined function on D, and it therefore makes sense to consider partial derivatives of $f_x$ with respect to $x$ and $y$. Such partial derivatives, if exist, are called second order partial derivatives of $f$, and are denoted respectively by $f_{xx}$ and $f_{xy}$. In fact, the subscripts indicate not only the orders of the partial derivatives but also the sequence in which they are taken, for examples, $f_{xx} = (f_x)_x$ and $f_{xy} = (f_x)_y$.

1. find the 1st order partail derivative of the following functions:

(i) $u = x^2 \sin\left(\frac{y}{x}\right)$ (ii) $u = \log(x^2 + y^2)$

sol. (i)

Here $u = x^2 \sin\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = 2x\sin(\frac{y}{x}) + x^2\cos(\frac{y}{x})(-\frac{y}{x^2})$$

$$= 2x\sin(\frac{y}{x}) - y\cos(\frac{y}{x})$$

$$\frac{\partial u}{\partial y} = x^2\cos(\frac{y}{x}).\frac{1}{x} = x\cos(\frac{y}{x})$$

(ii) here $u = \log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}.2x$$

$$= \frac{2x}{x^2 + y^2}$$

$$and \quad \frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}.2y$$

$$= \frac{2y}{x^2 + y^2}$$

4. If $u = e^{xyz}$, Prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

Sol.

$$u = e^{xyz}$$

Differentiating partially w.r.t. z

$$\frac{\partial u}{\partial z} = xye^{xyz}$$

Differentiating partially w.r.t. y

$$\frac{\partial^2 u}{\partial y \partial z} = xye^{xyz}.xz + xe^{xyz}$$

$$= (x + x^2yz)e^{xyz}$$

Again differentiating partially w.r.t. $x$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 2xyz)e^{xyz} + (x + x^2yz)e^{xyz}.yz$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

## Homogeneous functions:

- A function $f(x, y)$ of two variables $x$ & $y$ is said to be homogeneous function of degree n, if it can be expressed as

$$f(x,y) = x^n \phi_1\left(\frac{y}{x}\right) \; or \; y^n \phi_2\left(\frac{x}{y}\right)$$

Or

A function $f(x, y)$ is said to be homogeneous function of degree n, If $f(tx,ty) = t^n f(x,y)$.

- A function $f(x, y, z)$ of three variables $x, y$ & $z$ is said to be homogeneous function of degree n, if it can be expressed as

$$f(x,y,z) = x^n \phi_1\left(\frac{y}{x}, \frac{z}{x}\right) \; or \; y^n \phi_2\left(\frac{x}{y}, \frac{z}{y}\right) \; or \; z^n \phi_3\left(\frac{x}{z}, \frac{y}{z}\right)$$

e.g. (1) $f(x,y)=\dfrac{x^2+y^2}{x-y}$ is homogeneous of degree 1.

(2) $f(x,y)=\dfrac{x^2+y}{x-y}$ is not homogeneous function

## Euler's Theorem for homogeneous functions

**Statement:**

**If** $u$ **is homogeneous function of degree n in two variables** $x$ & $y$ **, then**

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

**Proof:**

Let $u = f(x, y)$

$$= x^n \phi\left(\frac{y}{x}\right) \quad .........(1) \qquad \{\text{As is homogeneous function of degree n}\}$$

Differentiating partially equation (1) w. r. t. $x$, we get

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\frac{-y}{x^2}$$

$$= nx^{n-1}\phi\left(\frac{y}{x}\right) - yx^{n-2}\phi'\left(\frac{y}{x}\right)$$

Multiply by $x$, we get

$$x\frac{\partial u}{\partial x} = nx^n \phi\left(\frac{y}{x}\right) - yx^{n-1}\phi'\left(\frac{y}{x}\right) \quad ................(2)$$

Again differentiating partially equation (1) w. r. t. $y$, we get

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1}\phi'\left(\frac{y}{x}\right)$$

Multiply by $y$ , we get

$$y\frac{\partial u}{\partial y} = yx^{n-1}\phi'\left(\frac{y}{x}\right) \qquad \text{................(3)}$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) = nu$$

Hence proved.

**If $u$ is homogeneous function of degree n in two variables $x$ & $y$, then**

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

**Proof:**

**As $u$ is homogeneous function of degree n in two variables $x$ & $y$, therefore by Euler's theorem**

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad \text{.................(1)}$$

Differentiating partially equation (1) w. r. t. $x$, **we get**

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

or
$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$$

Multiply by $x$, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=(n-1)x\frac{\partial u}{\partial x} \qquad \text{..............(2)}$$

Again Differentiating partially equation (1) w. r. t. $y$, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}$$

or
$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$$

Multiply by $y$, we get

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)y\frac{\partial u}{\partial y} \qquad \text{...............(3)}$$

Adding equations (2) and (3), we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)\left( x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) \qquad \left\{ \because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right\}$$

$$= n(n-1)u \qquad \{ \text{ using equation (1)} \}$$

Hence proved.

## *Composite Functions*

If $u = f(x, y)$, where $x = \phi_1(t)$, $y = \phi_2(t)$, then $u$ is called a composite function of the **single variable** $t$ and we can obtain $\frac{du}{dt}$ which is called the total derivative of $u$.

If $u = f(x, y)$, where $x = \phi_1(r, s)$, $y = \phi_2(r, s)$, then $u$ is called a composite function of **two variables** $r$ and $s$ and we can obtain $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.

*Differentiation of Composite Functions:*

A. If $u$ is a composite function of $t$, defined by the relation $u = f(x, y)$ and $x = \phi_1(t)$, $y = \phi_2(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

B. If $u$ is a composite function of $t$, defined by the relation $u = f(x, y, z)$ and $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$

C. If $u$ is a composite function of $r$ and $s$, defined by the relation $u = f(x, y)$ and $x = \phi_1(r, s)$, $y = \phi_2(r, s)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

D. If $u = f(x, y)$, where $y = \phi(x)$, then as $u = \psi(x)$, therefore $u$ is a composite functions of $x$, so we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

E. If $u = f(x, y)$, where $x = \phi(y)$, then as $u = \psi(y)$, therefore $u$ is a composite functions of $y$, so we have

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dy} + \frac{\partial u}{\partial y}$$

# MULTIPLE INTEGRALS

IMPORTANT TERMS, DEFINITIONS & RESULTS

## *09. Evaluation of a double integral*:

A double integral can be evaluated by successive single integrations i.e. as a two-fold iterated (repeated) integral as follows (if $R$ is regular in $y$-direction):

$$I_R = \int_{x=a}^{x=b} \left\{ \int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy \right\} dx \qquad (1)$$

Where the integration is performed first with respect to $y$ (within the braces). With the substitution of the limits $y_1(x)$ and $y_2(x)$, the integrand becomes a function of $x$ alone, which is then integrated with respect to $x$ from $a$ to $b$.

In a similar way, for a domain $R$ (regular in $x$-direction) which is bounded above by $x = x_2(y)$ and bounded below by $x = x_1(y)$ and the abscissa $y = d$ and $y = e$. The double integral is evaluated as

$$I_R = \int_{y=d}^{y=e} \left\{ \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx \right\} dy \qquad (2)$$

In this case the integration is first performed with respect to $x$ and then later with respect to $y$.

## *10. Change of order of integration:*

As already discussed, for the double integral with variable limits

$$I_R = \iint_R f(x, y)\,ds \qquad (1)$$

The limits of integration can be fixed from a rough sketch of the domain on integration. Then (1) can be evaluated as a two-fold iterated integral using either

$$I_R = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\,dydx \qquad (2)$$

or

$$I_R = \int_d^e \int_{x_1(y)}^{x_2(y)} f(x, y)\,dxdy \qquad (3)$$

In each specific problem, depending upon the type of the domain *R* and / or the nature of the integrand, choose either (2) or (3) whichever is easier to evaluate. Thus, in several problems, the evaluation of double integral becomes easier with the change of order of integration, which of course, changes the limits of integration also.

## 11. *General change of variables in Double Integrals:*

In several cases, the evaluation of double integrals becomes easy when we change variables.

Let $R$ be region in $xy$-plane and let $x, y$ be the rectangular Cartesian coordinates of any point $P$ in $R$. Let $u, v$ be new variables in region $R^*$ such that $x, y$ and $u, v$ are connected through the continuous functions (transformations)

$$x = g(u,v), \quad y = h(u,v)$$

Then $u, v$ are said to be curvilinear coordinates of point $P^*$ in $R^*$ which uniquely corresponds to $P$ in $R$. Then, a given double integral in the old variables $x$ and $y$ can be transformed to a double integral in terms of new variables $u, v$ as follows:

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} F(u,v)\,|J|\,dudv$$

Where $J$ is Jacobian defined as $J = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$

## ❖ *Double Integrals in Polar coordinates:*

For a double integral in Cartesian coordinates *x, y;* the change of variables to polar coordinates $r, \theta$ can be done through the transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

The Jacobian in this case is $J = J\left(\dfrac{x, y}{r, \theta}\right) = \dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

Therefore, the double integral in Cartesian coordinates *x, y* gets transformed to double integral in polar coordinated as follows:

$$\iint_R f(x, y)\,dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\;drd\theta$$

Where $R^*$ is the corresponding domain in polar coordinates.

### 14. *General change of variables in Triple Integrals:*

Let the functions $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$ be the transformations from cartesian coordinates $x$, $y$, $z$ to the curvilinear coordinates $u$, $v$, $w$. Let $F(x,y,z)$ be a continuous function defined in a domain $V$ in the $xyz$ coordinate system. Then a triple integral in cartesian coordinates $x$, $y$, $z$ can be transferred to a triple integral in the curvilinear coordinates $u$, $v$, $w$ as follows:

$$\iiint_V F(x,y,z)\,dx\,dy\,dz = \iiint_{V^*} G(u,v,w)\,|J|\,du\,dv\,dw$$

Where $J$ is Jacobian defined as $J\left(\dfrac{x,y,z}{u,v,w}\right) = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$ and $V^*$ is the corresponding domain in the curvilinear coordinates $u$, $v$, $w$

❖ ***Triple Integral in Cylindrical Coordinates:***

Cylindrical coordinates $r, \theta, z$ are particularly useful in problems of solids having axis of symmetry. The transformation of Cartesian coordinates $x, y, z$ in term of cylindrical coordinates is given by

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$ and the Jacobian in this case is given by

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Therefore, $\iiint\limits_{V} F(x,y,z)\,dx\,dy\,dz = \iiint\limits_{V^*} F(r\cos\theta, r\sin\theta, z)\,|J|\,dr\,d\theta\,dz$

$$= \iiint\limits_{V^*} F(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

## ❖ *Triple Integral in Spherical Polar Coordinates:*

In problems having symmetry with respect to a point O (generally the origin), it would be convenient to use spherical coordinates with this point chosen as origin. Coordinate transformation from $x, y, z$ to the spherical coordinates $r, \theta, \phi$ are given by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and the Jacobian in this case is given by

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & -r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Thus, $\iiint_V F(x,y,z)\,dx\,dy\,dz = \iiint_{V'} F(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$

$$= \iiint_{V'} F(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

## *15. Volume of a solid:*

Volume of a solid contained in the domain $V$ is given by the triple integral as follows:

$$V = \iiint_V dV = \iiint_V dx\, dy\, dz$$

# Infinite Series

## *n*th-Term Test

The *n*th-Term Test is also called the Divergence Test.

The *n*th-Term Condition is given below :

$$\lim_{n \to \infty} a_n \neq \infty$$

Note : This test can be used only for divergence.

This test cannot be used for convergence.

Basically, it says that, for a series $\lim_{n \to \infty} a_n$

if *an* ≠0, then the series diverges

# *P*-Series

The *p*-series is a pretty straight-forward series to understand and use.

The P-Series $\sum_{n=1}^{\infty} 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

Converges when $P > 1$

Diverges when $0 < p \leq 1$

# The P Series Convergence Theorem

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Converges when $P > 1$

Diverges when $0 < p \leq 1$

**In Summary:** For the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1. $p > 1$ **converges** by the integral test
2. $0 < p \leq 1$ **diverges** by the nth-term test
3. $p = 0$ **diverges** by the nth-term test
4. $p < 0$ **diverges** by the nth-term test

## p-series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges if $p > 1$, diverges if $p \le 1$.

We could show this with the integral test.

If this test seems backward after the ratio and nth root tests, remember that larger values of $p$ would make the denominators increase faster and the terms decrease faster.

$\rightarrow$

# Limit Comparison Test

If $a_n > 0$ and $b_n > 0$ for all $n \geq N$ ($N$ a positive integer)

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad 0 < c < \infty$ , then both $\sum a_n$ and $\sum b_n$ converge or both diverge.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ , then $\sum a_n$ converges if $\sum b_n$ converges.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ , then $\sum a_n$ diverges if $\sum b_n$ diverges.

$\rightarrow$

# D `Alembert's Ratio Test:

If $\sum a_n$ is a series with positive terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

then:

The series converges if $L < 1$ .

The series diverges if $L > 1$ .

The test is inconclusive if $L = 1$.

$\rightarrow$

# Raabe's Test

Let $\sum a_n$ a series of positive terms and let limit n[un/un+1 – 1]=l. Then

(a) if l>1, $\sum a_n$ converges

(a) if l<1, $\sum a_n$ diverges

(a) the test fails when l=1.

## Cauchy's Root Test:

If $\sum a_n$ is a series with positive terms and $\lim_{n\to\infty} \sqrt[n]{a_n} = L$

then:

The series converges if $L < 1$ .

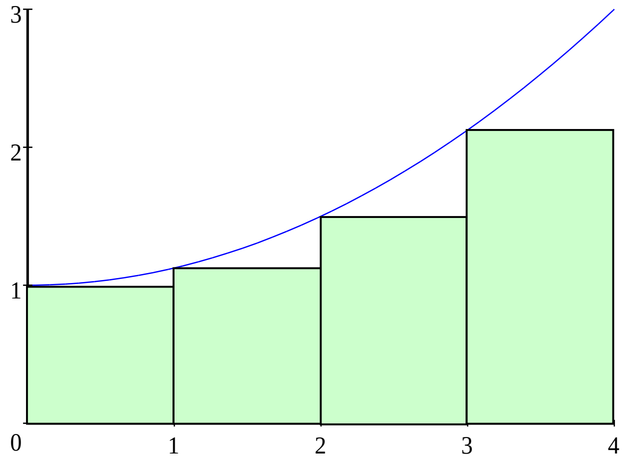
The series diverges if $L > 1$ .

The test is inconclusive if $L = 1$.

Note that the rules are the same as for the Ratio Test.

→

Remember that when we first studied integrals, we used a summation of rectangles to approximate the area under a curve:




This leads to:

## Cauchy's Integral Test

If $\{a_n\}$ is a positive sequence and $a_n = f(n)$ where $f(n)$ is a continuous, positive decreasing function, then:

$\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge .

# Alternating Series Test

The Alternating Series Test is sometimes called the Leibniz Test or the Leibniz Criterion.

$$\sum_{i=0}^{n} (-1)^n a_n$$

Converge if both of the following condition hold :

Condition 1 : $\lim_{n \to \infty} a_n = 0$

Condition 2 : $0 < a_{n+1} \leq a_n$

## Absolute And Conditional Convergence

*Sometimes a series will have positive and negative terms, but not necessarily alternate with each term. To determine the convergence of the series we will look at the convergence of the absolute value of that series.*

Absolute Convergence: If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges

Conditionally Convergent: If $\sum a_n$ converges but $\sum |a_n|$ diverges.