# Module 2: Time Response Analysis.

## Initial value theorem :-

If the Laplace transform of $f(t)$ is $F(s)$, then

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

if the limit exists.

## Final Value theorem :-

If the Laplace transform of $f(t)$ is $F(s)$ and if $sF(s)$ is analytic on the imaginary axis and in the right half of the s-plane then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} F(s)$$

The final-value theorem is very useful for the analysis and design of control systems. Since, it gives the final value of a time function by knowing the behaviour of its laplace transform at $s=0$.

* The final value theorem is not applicable/valid if $sF(s)$ contains any pole whose real part is zero or positive, which is equivalent to the requirement of $sF(s)$ in the right-side of the plane.

Automatic Control Systems
Benjamin C. Kuo [pg.No - 19-20].

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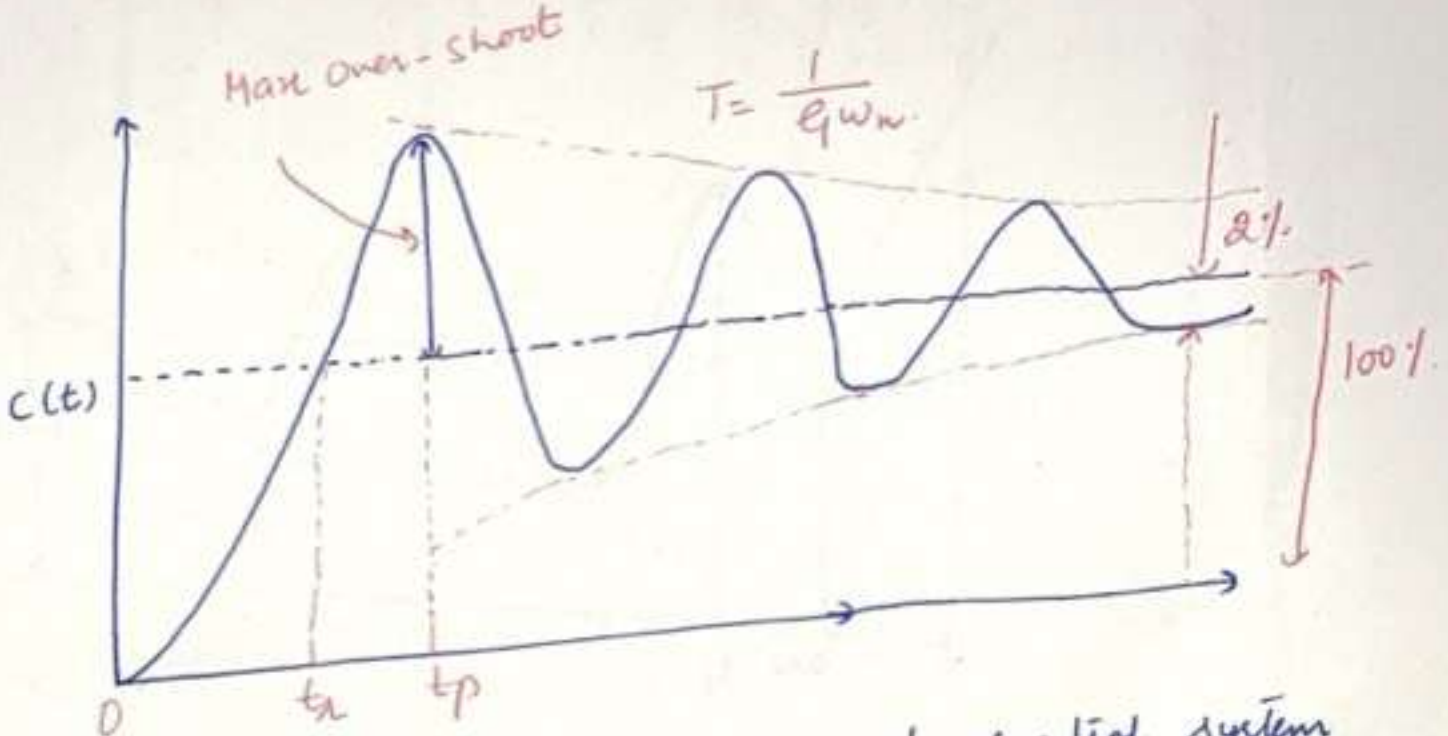
① Transient Response specification of second order control system




The time response of an underdamped control system exhibits damped oscillations prior to reaching steady state. The specifications pertaining to time response during transient part are shown.

(a) The Rise time : $t_r$

The rise time is the time needed for the response to reach from 10% to 90% or 0 to 100% of the desired value of the output at the very fast instant.

Usually 0 - 100% basis is used for underdamped system and
10 - 90% " " " overdamped system

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\, t + \phi\right]$$

At the first instant when time response reaches 100% of the desired value i.e. $c(t)=1$ the time is $t_r$

substituting $c(t) = 1$

$$1 = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2}\, t_r + \phi\right].$$

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$$\frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n \sqrt{1-\xi^2}) t_r + \phi\right] = 0.$$

As $\dfrac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}}$ is finite

$$\sin\left[(\omega_n \sqrt{1-\xi^2}) t_r + \phi\right] = 0$$

The acceptable solution of the above eqn is

$$\left[(\omega_n \sqrt{1-\xi^2}) t_r + \phi\right] = \pi$$

so $t_r = \dfrac{\pi - \phi}{\omega_n \sqrt{1-\xi^2}}$ where $\phi = \tan^{-1}\left[\dfrac{\sqrt{1-\xi^2}}{\xi}\right]$.

(b) Maximum Overshoot : $M_p$
and peak time : $t_p$

The maximum positive deviation of the o/p w.r.t its desired Value is Known as maximum overshoot and denoted as $M_p$.

$$M_p = c(t)_{max} - 1$$

$$\% M_p = \frac{c(t)_{max} - 1}{1} \times 100$$

- The time needed to reach the max overshoot is called peak time. denoted as $t_p$.
- $$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n \sqrt{1-\xi^2}) t + \phi\right]$$

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$$\frac{dc(t)}{dt} = -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cdot \omega_n \sqrt{1-\xi^2} \cdot \cos[(\omega_n \sqrt{1-\xi^2})t + \phi]$$

$$- \frac{(-\xi\omega_n) e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[(\omega_n \sqrt{1-\xi^2} t + \phi]$$

put $\frac{dc(t)}{dt} = 0$

$$\therefore \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} [-\omega_n \sqrt{1-\xi^2} \cdot \cos[(\omega_n \sqrt{1-\xi^2} t + \phi]$$

$$+ \xi\omega_n \sin[(\omega_n \sqrt{1-\xi^2})t + \phi] = 0$$

In the above eqn, since $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$ is finite

$$\therefore \omega_n \sqrt{1-\xi^2} \cdot \cos[(\omega_n \sqrt{1-\xi^2} t + \phi] = \xi\omega_n \sin[(\omega_n \sqrt{1-\xi^2} t + \phi]$$

$$\therefore \tan[(\omega_n \sqrt{1-\xi^2})t + \phi] = \frac{\sqrt{1-\xi^2}}{\xi}$$

Since $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$.

$$\tan\left[(\omega_n \sqrt{1-\xi^2})t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right] = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$(\omega_n \sqrt{1-\xi^2})t = n\pi$$

where $n = 0, 1, 2 \ldots$

The instant of occuring max overshoot is identified by putting $n=1$

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

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$c(t)_{max}$ is determined by putting $t = t_p$

(1)

$$c(t)_{max} = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \cdot \sin\left[(\omega_n\sqrt{1-\xi^2})\, t_p + \phi\right]$$

$$= 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot \sin\left[(\omega_n\sqrt{1-\xi^2}\,\frac{\pi}{\omega_n\sqrt{1-\xi^2}} + \phi\right]$$

$$= 1 - \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \cdot \sin(\pi + \phi) = 1 - \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \cdot -\sin\phi$$

$$\because \quad \phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}, \quad \therefore \sin\phi = \sqrt{1-\xi^2}$$

$$\cancel{c(t)=}\; c(t)_{max} = 1 + \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \cdot \sqrt{1-\xi^2}$$

$$c(t)_{max} = 1 + e^{\xi\pi/\sqrt{1-\xi^2}}$$

or

$$M_p = c(t)_{max} - 1$$

$$\therefore = 1 + e^{-\xi\pi/\sqrt{1-\xi^2}} - 1$$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$$

$$\%\, M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100.$$

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(3) (C) settling time ts.

the time needed to settle down aforesaid oscillations within 2% of desired value of the o/p is known as settling time and denoted as ts.

$$t_s = 4 \frac{1}{\xi \omega_n}.$$

On 5% basis, the settling time for a second order control system is approx three times the time constt.

$$t_s = 3 \frac{1}{\xi \omega_n}.$$

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- Any System containing energy storing element like inductor, capacitor, mass and inertia etc posses certain energy. These energy storage elements are part of the control system and cannot be avoided.

- If the energy state of the system is disturbed then it takes a certain time to change from one state to another state. This disturbance sometimes occurs at input. Sometimes occurs at output and sometimes at both ends. The time required to change from one state to another state is known as transient time. and the values of current and voltages during this period is called transient response. ~~These~~

Disturbance

- These transients may have oscillations ~~which~~ may be either sustained or decaying in nature. This will depend upon the parameters of the system.

equal area criteria

- For any system, we obtain a linear differential equation. The solution of linear differential equation gives the response of ~~that~~ the system] In time domain

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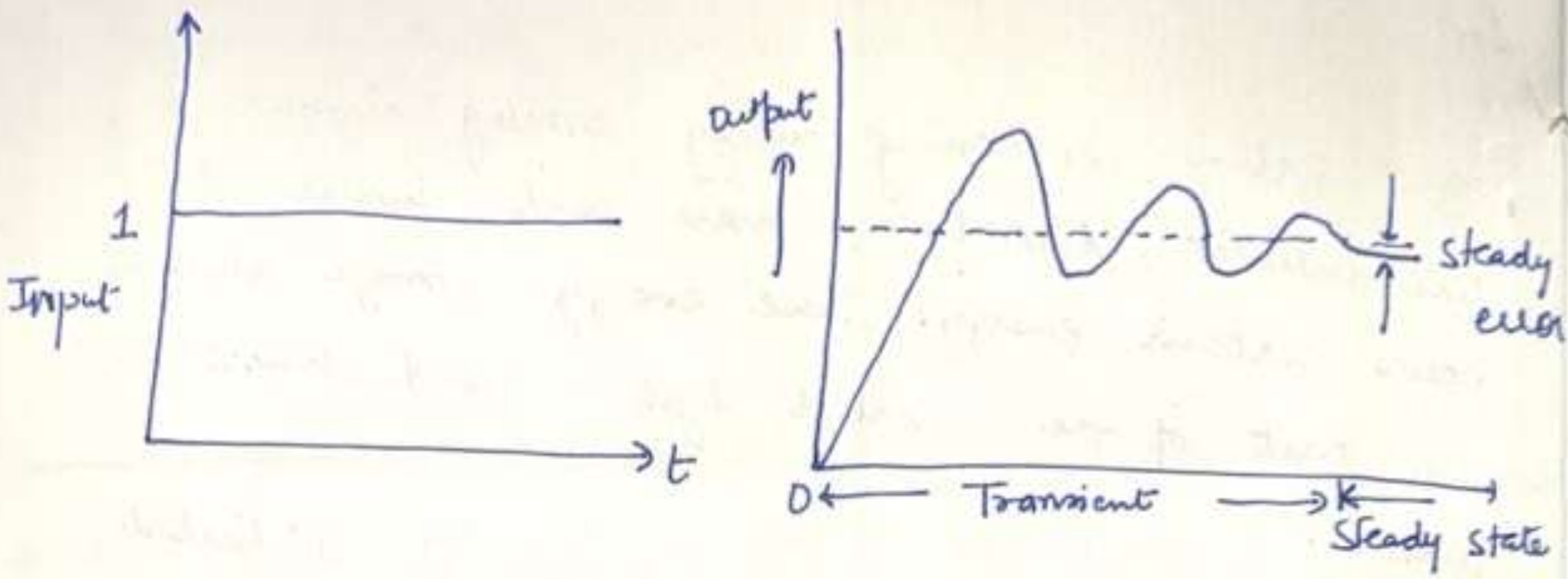


It is clear that the transient response is the part the response which goes to zero as time increases and steady state response is the part of the total response after transient has died.

If the steady state ~~error~~ response of the output doe not match the input then the system has steady sta error.

- The transient part of time the response reveals the nature of response (i.e. oscillatory or overdamped) an also gives an indication about its speed.
- The steady state part of time response reveals the accuracy of a control system. steady state error is observed if the actual output does not exactly match with the input.

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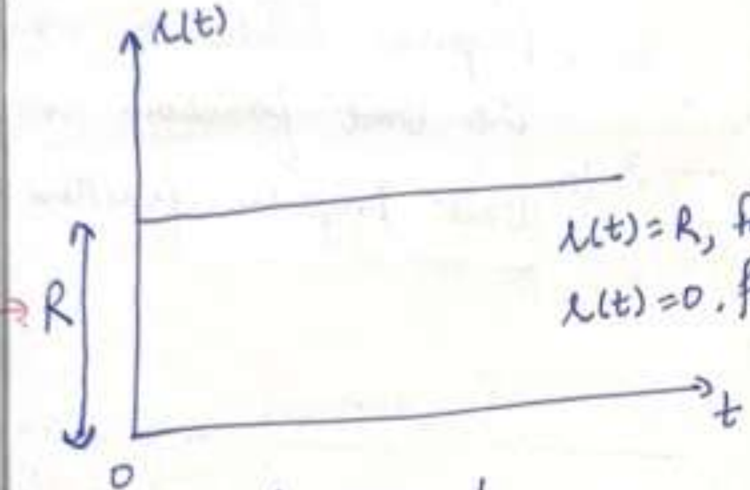
## Input test signals :-

(i) step function :

Sudden application of input signal




$r(t) = R$, for $t \geq 0$
$r(t) = 0$, for $t < 0$

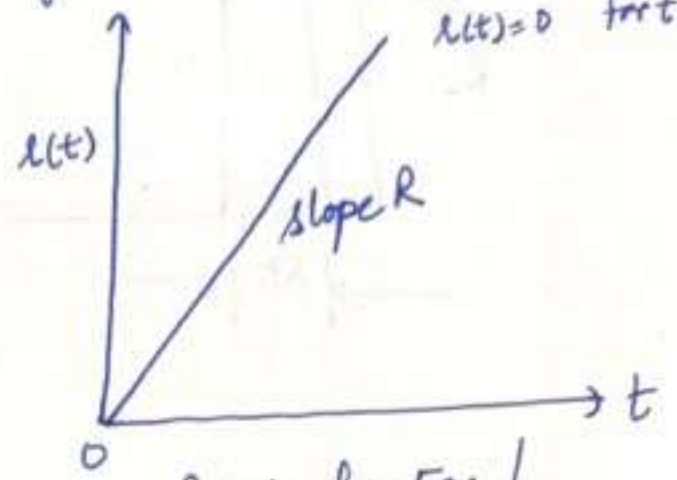
step function / displacement function

if R = 1 unit,
the step function is called
Unit step function

$$L(1) = \frac{1}{s}$$

(ii) Ramp function

$r(t) = Rt$ for $t \geq 0$
$r(t) = 0$ for $t < 0$




Ramp function / velocity function

gradual application of I/P.

if R = 1, [Unit ramp function]

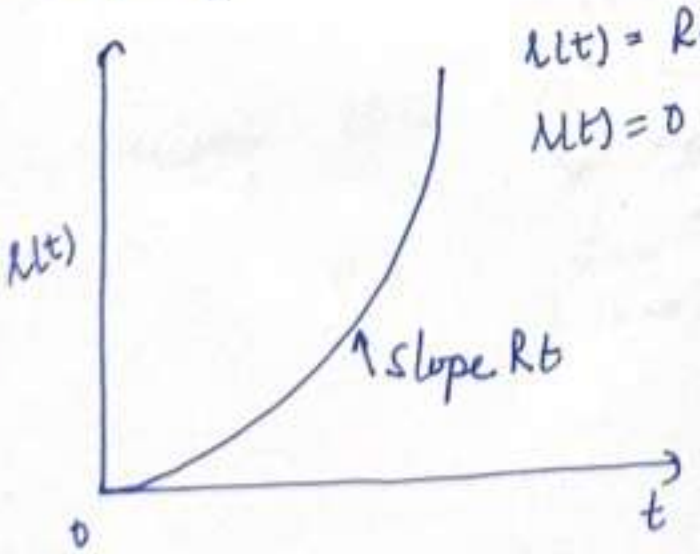
$r(t) = t$

$$L(t) = \frac{1}{s^2}$$

(III) Parabolic function

more gradual application of I/P in comparison with ramp function

$r(t) = Rt^2/2$ for $t \geq 0$
$r(t) = 0$ for $t < 0$




if R = 1, then

$$r(t) = \frac{t^2}{2}$$

and parabolic function is called unit parabolic function

$$\mathcal{L}\left(\frac{t^2}{2}\right) = \frac{1}{s^3}$$

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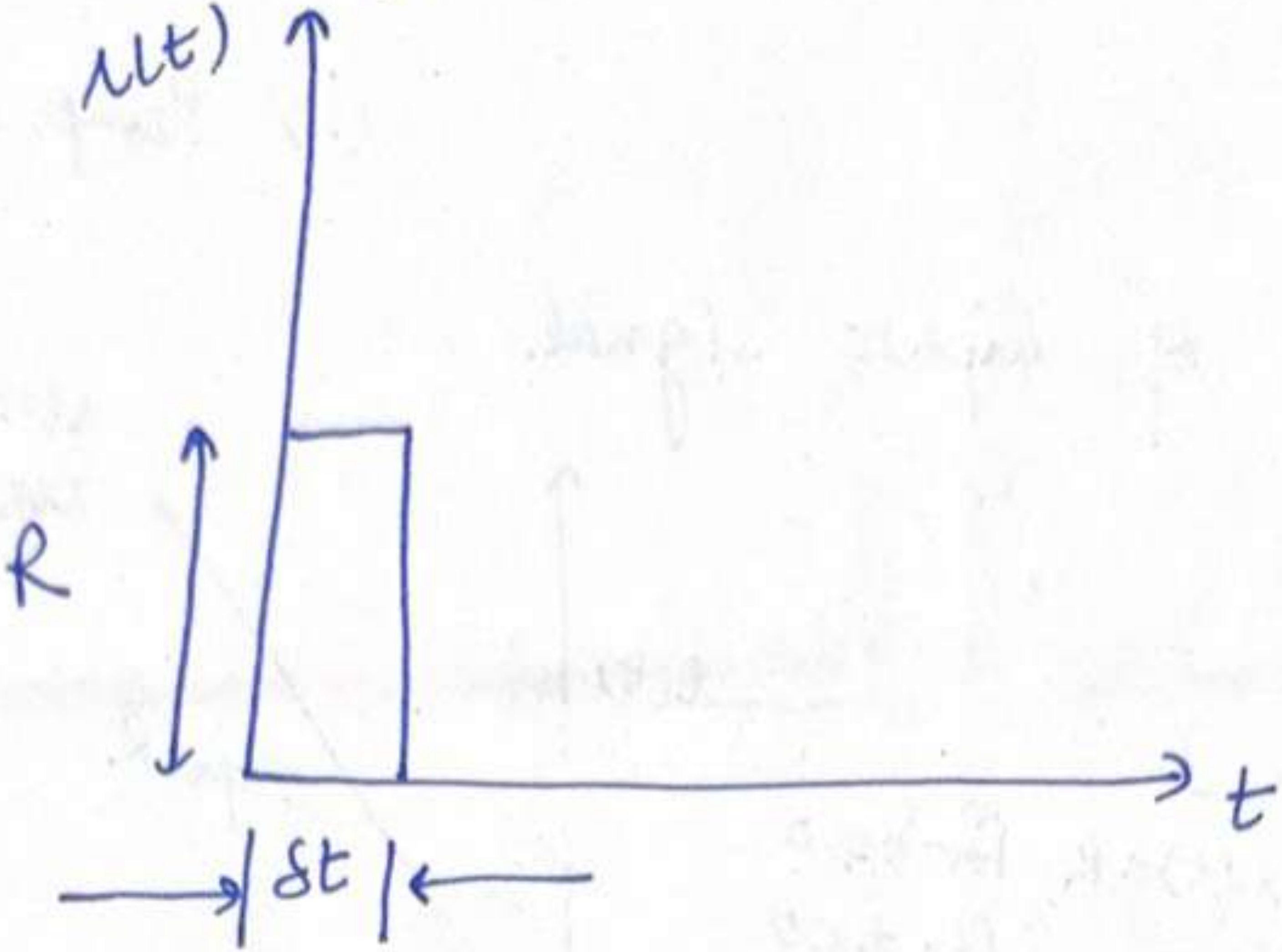
(iv) Impulse function :




input is applied [...]
shock for a ve[...]
duration of time. [...]

If the magnitude [...]
impulse function is [...]
the unit function i[...]
Unit impulse function [...]

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# Module 2 : Time Response Analysis.

## Design specifications for second order Control systems based on time Response

A Control system requires to meet three response specifications :

1. Steady state accuracy [permissible error $e_{ss}$]
   it is met by suitable choice of $K_p$, $K_v$ or $K_a$.
2. damping factor $\xi$ [peak overshoot to step I/P $M_p$ [5-40%]]
   it is preferred less than 1 in most of the control systems
   Range is [0.7-0.28].
3. settling time $t_s$.

Rise time $t_r$ is also specified. it shd be consistent with the specification of $t_s$

* $K_p \rightarrow$ Static positional error coeff. (Unit step I/P applied)
  $K_v \rightarrow$ " Velocity " " ( " Ramp " )
  $K_a \rightarrow$ " Acceleration " " ( " Parabolic " )

Definitions are on Pg. NO-155 (B.S. Manke)

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| M | | 4 | 11 | 18 | 25 |
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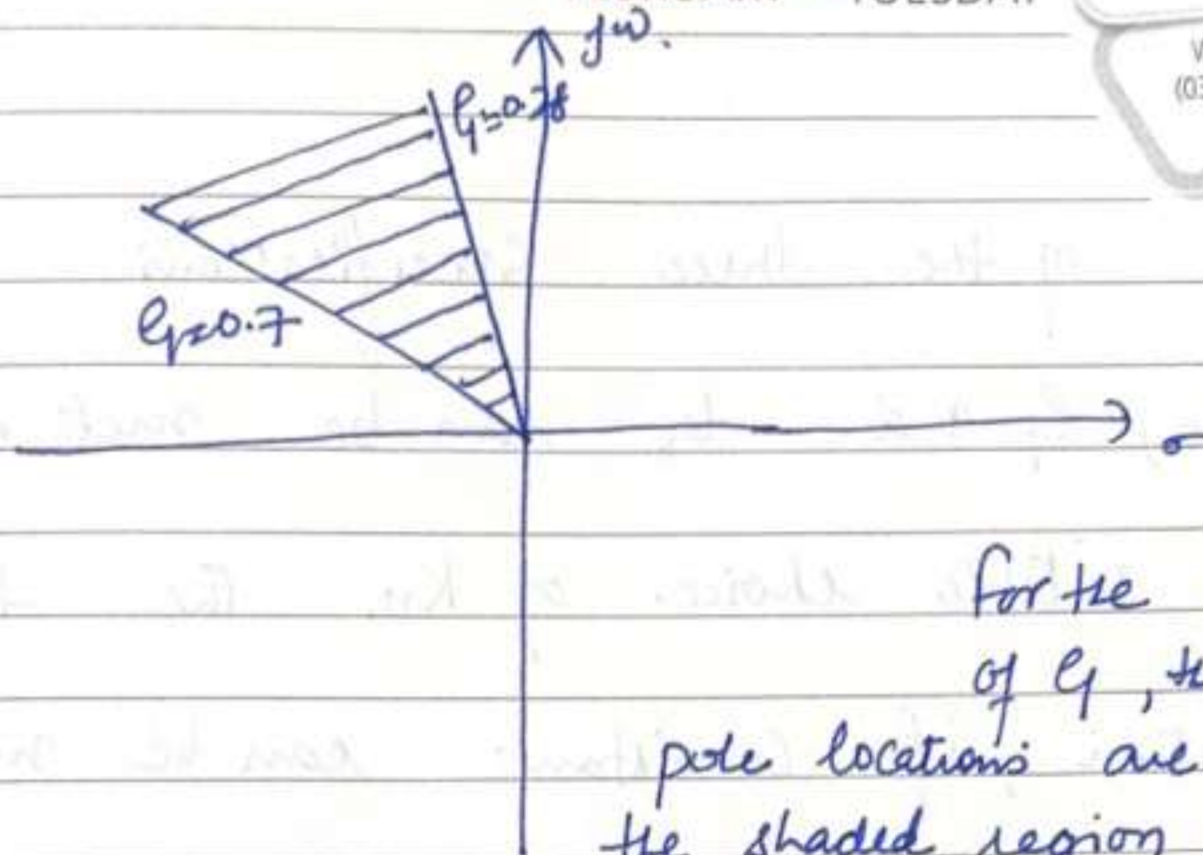
| T | | 7 | 14 | 21 | 28 |
| F | 1 | 8 | 15 | 22 | 29 |
| S | 2 | 9 | 16 | 23 | 30 |
| S | 3 | 10 | 17 | 24 | 31 |






for the given range of $\zeta$, the closed loop pole locations are restricted to the shaded region of the S-plane.

Examine the expressions for $e_{ss}$, $\omega_n$, $\zeta$ and $t_s$ for a type - 1 second order system

$$\zeta = \frac{1}{2\sqrt{K_v \tau}} \qquad \omega_n = \sqrt{K_v/\tau}.$$

$$t_s = \frac{4}{\zeta\omega_n} \quad (\text{for } 2\% \text{ tolerance band})$$

$$e_{ss} = \frac{2\zeta}{\omega_n} = \frac{1}{\omega_n} \quad (\text{for unit-ramp o/p})$$

Open loop TF :- Product of $G(s)$ [Forward path T.F] and $H(s)$ [feedback path TF]

So, $$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\cdots}{s^N(1+sT_1)(1+sT_2)\cdots} \qquad K \to \text{forward path gain}$$

$-\frac{1}{T_a}, -\frac{1}{T_b} \ldots$ are zeros and $-\frac{1}{T_1}, -\frac{1}{T_2} \ldots$ are poles and N is no of poles at origin.

if N=0 for type 0 sys. N=1 for type 1 and so on.

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Only two of the three specifications ie $e_{ss}$, $\zeta$ and $t_s$ can be met exactly by a suitable choice of $K_v$. The third specification, if consistent can be met as an upper or lower bound. [$\tau$ is generally fixed).

★ $K_v$ or open loop gain is the only adjustable parameter of the system [by varying amplifier gain).

Second order system can meet only one of the specifications exactly and this specification is on the allowable steady state error.



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So, to meet two independent specifications

9 a second order ~~s/s~~ system requires to be modified.

10 This modification is termed as 'compensation'

11

This compensation shd allow for high ~~static~~

12 open-loop gains to meet the specified steady

1 state accuracy and yet preserve a

2 satisfactory dynamic performance.

3 Some practical modification techniques are:

4 Derivative error, compensation

5 Derivative output compensation

6 Integral error compensation

Proportional plus integral plus derivative controller (PID)



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Concept of stability :- Routh Hurwitz Criterion

Relative stability analysis.

$$G(s) = \frac{A(s)}{B(s)} = \frac{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n}{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m}$$

After factorization

$$G(s) = \frac{A(s)}{B(s)} = \frac{K(s-s_1)(s-s_2)\dots(s-s_n)}{(s-s_a)(s-s_b)\dots(s-s_m)}$$

$K = \frac{a_0}{b_0}$ is known as gain factor of the TF.

if s is put equals to $s_a, s_b \dots s_m$, the value of TF is infinite, hence $s_a, s_b \dots s_m$ are called poles of the TF.

If s is put equals to $s_1, s_2 \dots s_n$, the value of TF is zero. hence $s_1, s_2 \dots s_n$ are called zeros of the TF.

The poles and zeros are either real or complex. and complex poles or zero always appear in conjugate pairs.



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* poles and zeros may coincide such poles and zeros are known as multiple poles or zeros.

* If the order of characteristic equation of a system is upto 2, the time solution of the system is conveniently obtainable and from the time solution the stability and instability can be obtained.
But if the order of the system is higher, stability of such a system cannot be easily determined by direct solution.

## Definition of stability :-

If any oscillations set up in the system in consequence to application of an input are damped out with respect to time, the system is said to be stable. and for unstable system, oscillations are increasing in magnitude. Also, if the magnitude of the oscillations are sustained, the system is marginally stable.


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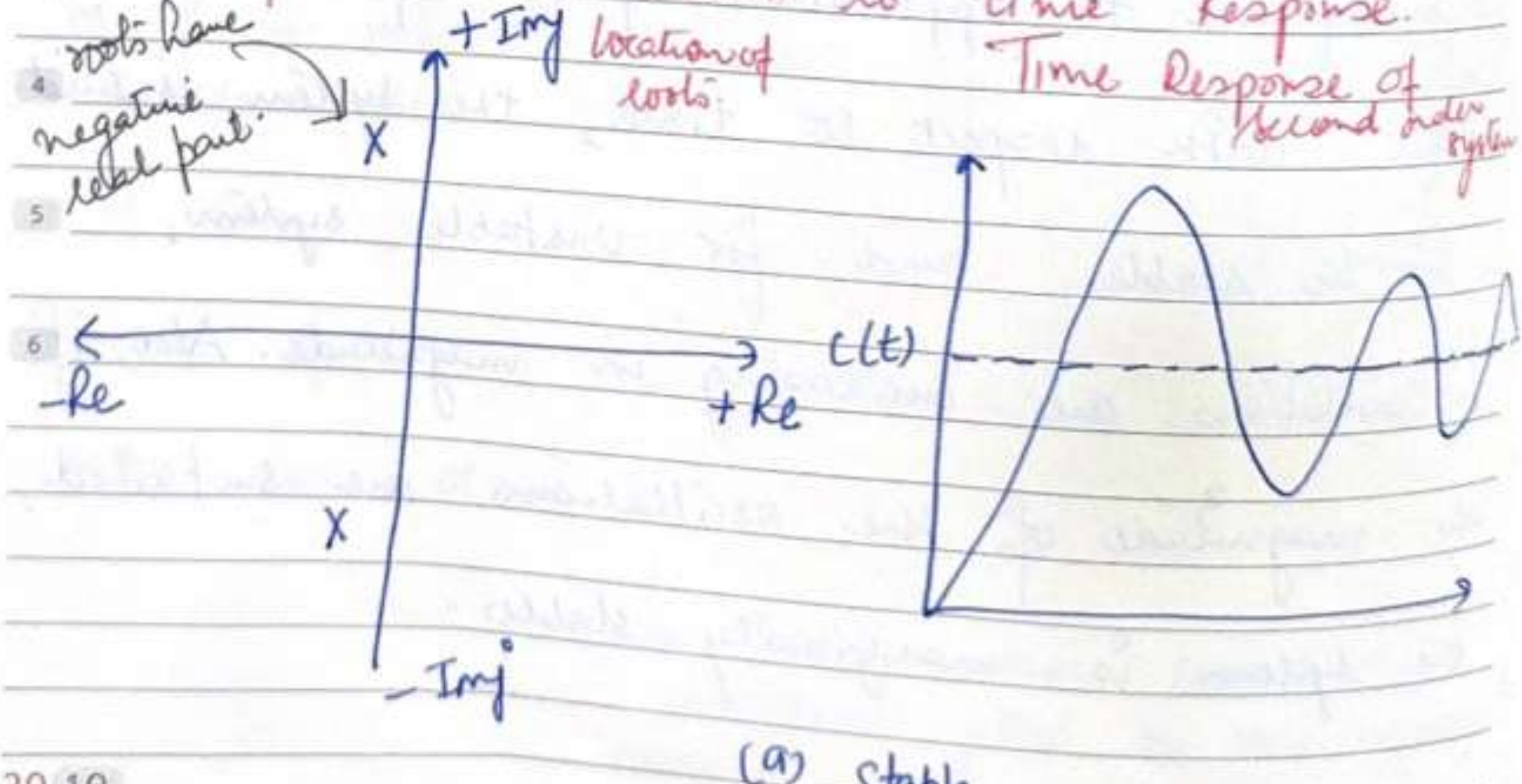
**Absolute Stability** :- is used in relation to qualitative analysis of stability. [can be determined from location of roots of the char equation in S-plane].

**Relative Stability** :- is used in relation to Comparative analysis of stability
[Maximum Overshoot, damping ratio, gain margin and phase margin are measures to relative stability].

**Location of roots of characteristic equations in S-plane as related to time Response.**




(a) Stable



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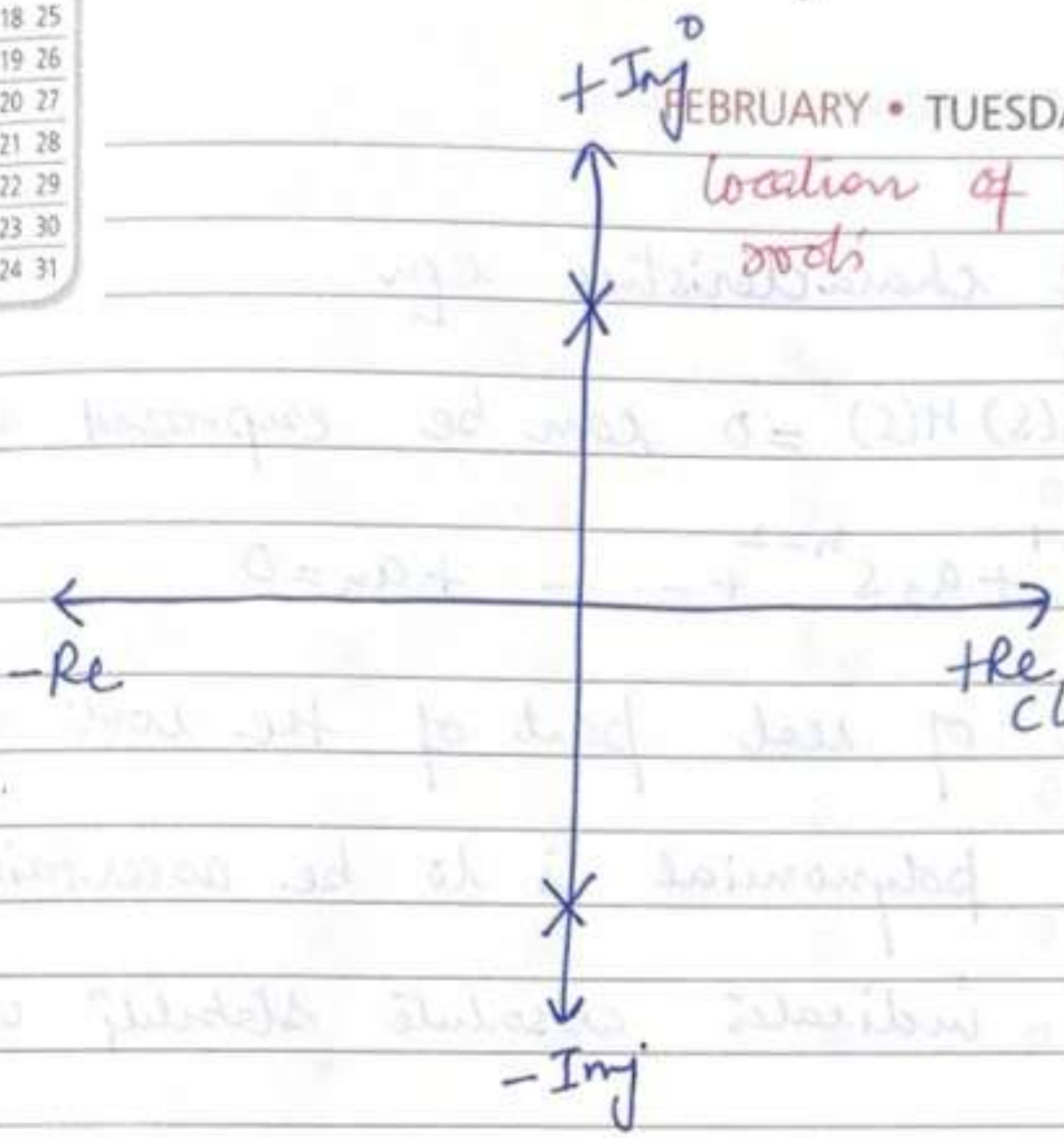
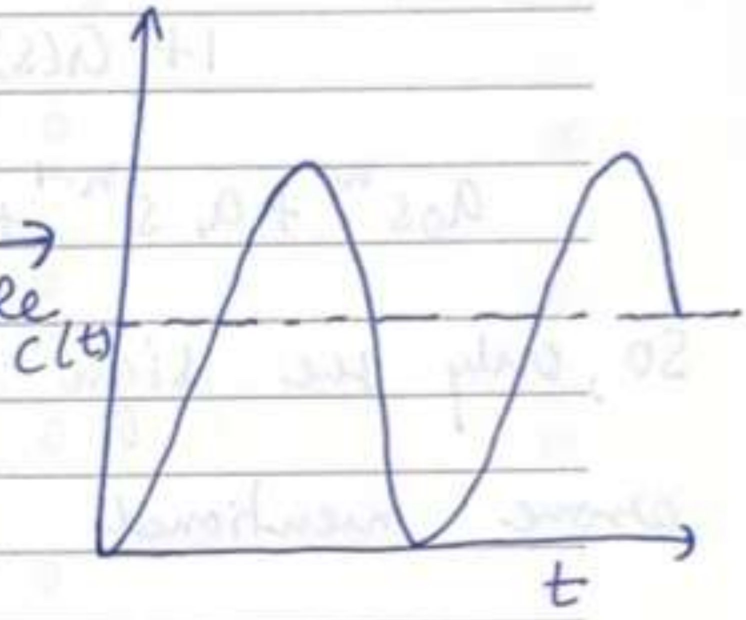
Absolute stability is determined by examining the sign of real parts of the roots of a system char. Eqn.

location of roots | Time Response.

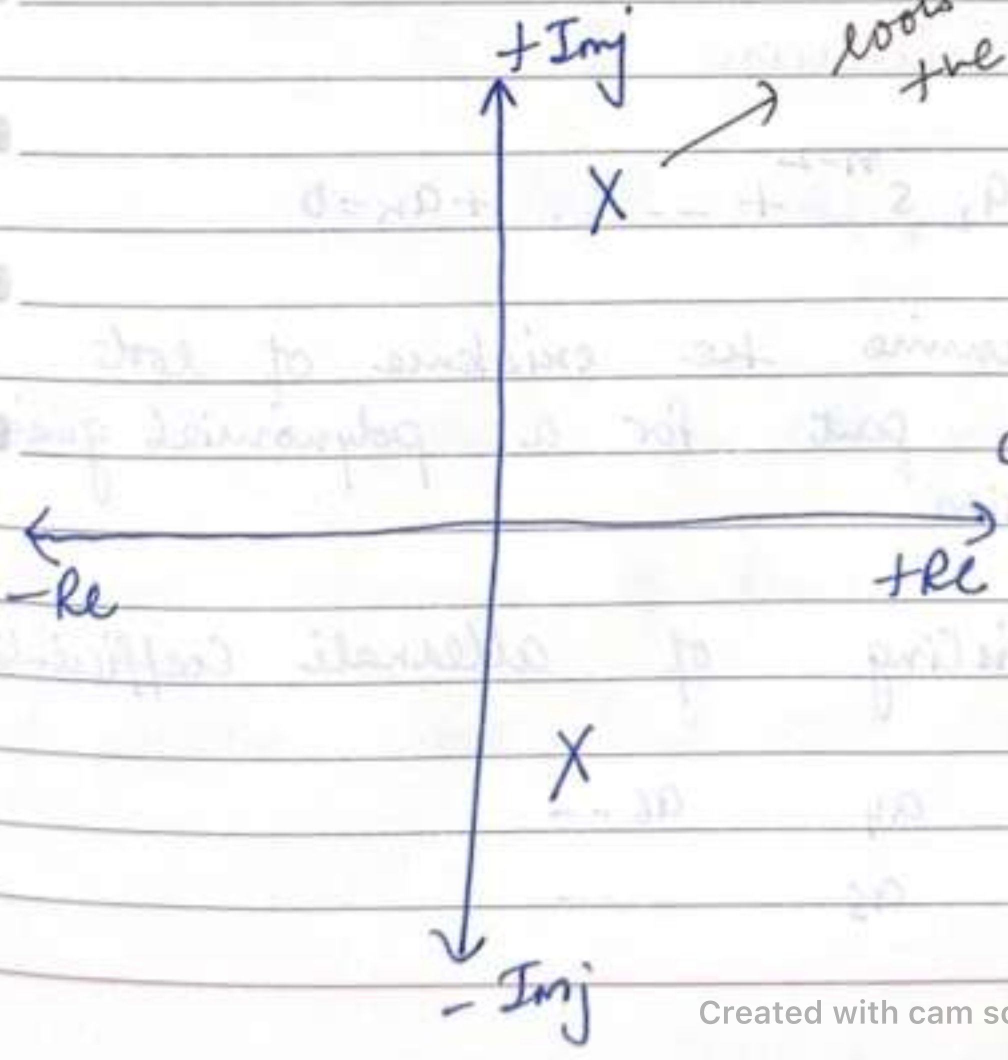


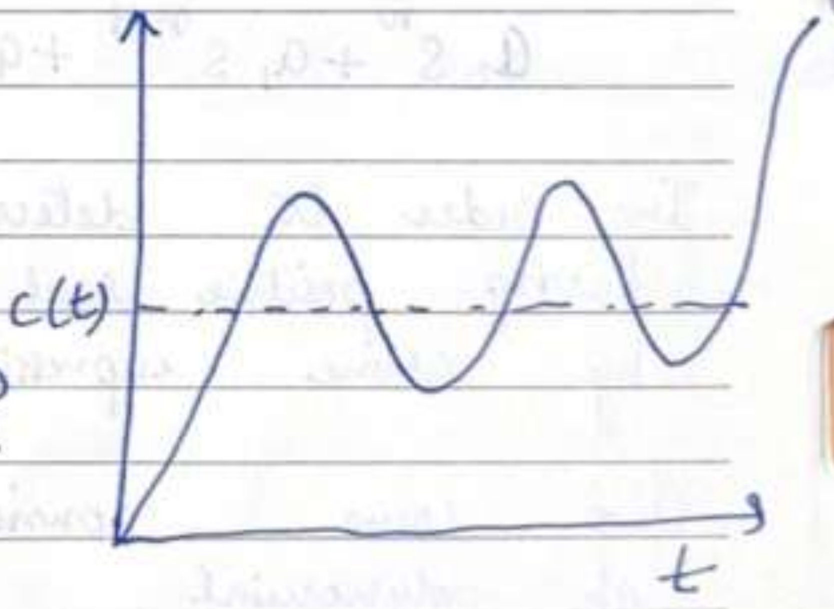




(b) (marginally stable)

even if one root is having a positive real part, the system is unstable.

roots have +ve real part



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## Relative Stability Analysis :-

Once a system is shown to be stable, we proceed to determine its relative stability by finding the settling time of the dominant roots of its char. eqn. The settling time being inversely proportional to the real part of the dominant roots, the relative stability can be specified by requiring that all the roots of the characteristics equation be more negative than a certain value.

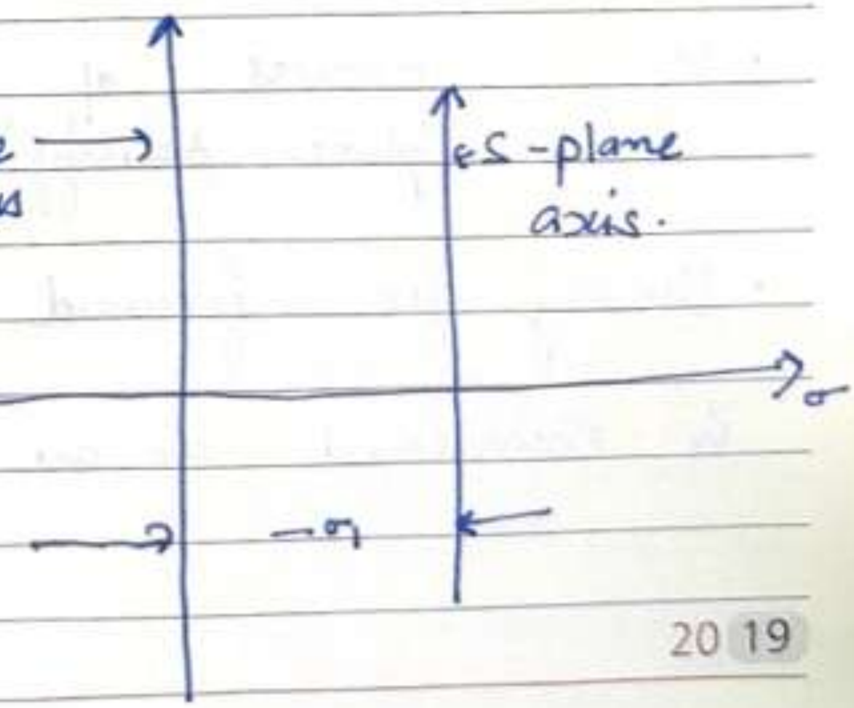
All the roots must lie to the left of the lines

$$s = -s_1 \ (s_1 > 0)$$

The characteristics eqn of the system under study is then modified by shifting the origin of the s-plane to $s = -\sigma_1$. by the substitution

$$s = z - s_1$$

If new char eqn in $z$ satisfies the Routh criterion It implies that all the roots of the original char eqn are more negative than $-\sigma_1$.



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